J. of Natural & Physical Sciences

Vol. 5 - 11 1991 - 1997

113222

स्थापना १९५९ विक्रमाब्द

खण्ड Volume 5 - 8 (1991 - 1994)

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका

JOURNAL OF NATURAL & PHYSICAL SCIENCES

गुरुकुल कांगड़ी विश्वविद्यालय, हरिद्वार
Gurukula Kangri Vishwavidyalaya, Hardwar

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Printed at :- Sadbhawna Printers (offset), F-22 Ind. Area Hardwar Phone 425751

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KIRAN OFFSET PRINTING PRESS, Kankhal, Haridwar. Ph.:415975

113222

113222

ISSN 0970-3799

खण्ड Volume 5-8 (1991-1994)

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका

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Gurukula Kangri Vishwavidyalaya, Hardwar

Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 1-10

'CHEMICAL INVESTIGATIONS OF SOME INDIGENOUS BASIL OILS'

Satinder Kaur* & Gopal K. Sinha*

(Received 30-01-1991)

ABSTRACT

Chemical composition of essential oil of *Ocimum sanctum* Linn distributed in and around Rishikesh and from the plants raised at Haldwani from the seeds of the plants distributed at Rishikesh, have been studied by derivatisation, TLC, GLC, PMR, MS and IR. In all twenty compounds have been identified as against nine compounds reported in literature, Principal components are : eugenol, methyl eugenol, and methyl chavicol. The Rishikesh oil sample containing 66.5 percent eugenol can serve as a substitute for oil of cloves.

Keywords and Phrases : Essential oil, *Ocimum sanctum*, Chemistry-subject classification:Pharmaceutical analysis Essential Oils - 62.

INTRODUCTION

Ocimum sanctum, belonging to N.O. Labiateae, is widely distributed in the warmer parts of the world. In India it is commonly known as 'Sri tulsi' and 'Krishna tulsi' and holds promise due to its high eugenol content. It is held sacred and the plant and its parts are credited possessing numerous curative effects probably not shared by many other plants.

Essential oil from *O. sanctum* has not been thoroughly investigated. Only a few constituents have been reported viz. linalool, geraniol, cirtronellol, cineole and methyl chavicol in the oil of plants raised at Phillipines by Guenther [5]; Cineole, eugenol and methyl chavicol by Gildmeister et al. [3]; eugenol, methyl chavicol, carveol and caryophyllene by Datt et al. [2]; eugenol, methyl eugenol and caryophyllene by Lawrance et al. [12]; eugenol, methyl chavicol, linalool and 1:8 cineole by Satti et al. [15]; Gibberellic acid (GA_3) treatment of the crop of *O. sanctum* has been reported increasing the essential oil as well as eugenol content [6]; Harvesting the crop at full bloom

---

*Chemical Laboratories, Govt. P.G. College, Rishikesh

possessed maximum eugenol content [7] [1] [16] [18] and that the oil possessed larvicidal, synergistic and antibacterial/ antifungal properties [1] [17]. In India and Pakistan the oil is used as a substitute for oil of cloves [8].

Essential oils were extracted, separately, from the plants of *O. sanctum* distributed at Rishikesh and from the plants raised at Haldwani (another place on the foot hills of Himalayas) from the seeds of Rishikesh plants. Chemical composition of the two oil samples were studied in order to know the difference, if any, in their composition due to change of place of distribution/cultivation.

EXPERIMENTAL :

Essential oil was obtained by steam distillation of the fresh plant material. The oil was found mostly concentrated in leaves and flowers. Stalks contained negligible oil. Oil content in leaves and flowers at various phases of plant development are given in table.1.

TABLE 1

OIL CONTENT IN LEAVES & FLOWERS AT DIFFERENT STAGES OF PLANT DEVELOPMENT AT RISHIKESH

Stage of growth	part of the plant	oil content (fresh weight basis)
Pre flowering (immature inflorescence)	leaves	0.65%
	flowers	0.60%
Flowering	leaves	0.58%
	flowers	0.41%
Post flowering	leaves	0.56%
	flowers	0.41%

Physico-chemical properties of the essential oil obtained from the plants (flowering stage) distributed at Rishikesh were:

Sp. gravity/20° = 0.9498 ; Ref. index/20° = 1.5184 ;
Sp. rotation 20° = - 6° 20' ; Acid value = 1.2 ; Ester value = 24.1 ;
Carbonyl value as $C_{10} H_{16}0$ (Oximation) = 1.5 ;
Phenolic content = 66.2 and solubility in 80% alcohol = 1:0.60.

Chemical composition of the essential oil was studied by the usual derivatization, chromatographic processes like TLC & GLC and IR, UV, PMR and Mass spectroscopy. 50 ml of the oil was shaken thoroughly with 5% aqueous sodium hydroxide followed by 1% sodium hydroxide. The oily layer was separated from the reaction mixture by extraction with petroleum ether (40-60°) and the non-phenolic constituents (16.8 ml) were obtained by distilling off the solvent from the ether extract. The aqueous sodium hydroxide extract was acidified with 20% phosphoric acid to Congo red point and extracted with ether. Ethereal extract was dried over anhydrous sodium sulphate and on distilling off the solvent 32.5 ml phenolic contents were obtained. Constituents present in phenolic and non-phenolic fractions were investigated separately.

PHENOLIC FRACTION :

TLC examination over silica gel G layers under different solvent systems and GLC indicated this fraction to be a single compound. The fraction possessed $d_4^{20} = 1.0616$; $n^{20}_D = 1.541$ and formed phenyl urethane m 97°, benzoate m 69°and tetrabromide m 118° [10] [9] [4]. These properties and IR spectra confirmed this fraction to be eugenol.

NON-PHENOLIC FRACTION :

This fraction was subjected to further separation -

a) Separation of alcohols :

The alcohols from the non-phenolic fraction (16.8 ml) were separated by refluxing the fraction with calculated amounts of 3:5 Dinitro benzoyl chloride and pyridine. The reaction mixture was cooled and water was added. The solid derivative was separated and dried.

TLC examination of the solid derivative over silica gel G coated plates, using benzene : petroleum ether (1:1) as solvent system, butter yellow as reference compound and ethanolic α - naphthyl amine solution (1%) as spray reagent showed the presence of linalool ($R_B = 1.75$), α- terpineol ($R_B = 1.4$) and citronellol ($R_B = 1.65$). Identity of these compounds was further confirmed by Co - TLC.

b) Alcohol free non-phenolic fraction :

The fraction (12.5ml), left after distilling off the pyridine under reduced pressure, was separated quantitatively into 'Hydrocarbon fraction' and

'Oxygenated fraction' by using a column packed with silicic acid, and n-hexane and ethyl acetate, respectively, as eluting agents [11].

Hydrocarbon fraction :

TLC examination of the fraction over silica gel G coated plates, using n-hexane as solvent system, limonene as reference compound, fluorescein-bromine as reagent for visual identification showed the presence of three compounds viz. β - elemene (R_L = 1.3), β - caryophyllene (R_L = 1.4) and terpinolene (R_L = 1.5). Identity of these compounds was confirmed by Co-TLC, and PMR/Mass spectroscopy of the compounds obtained from preparative chromatography of the 'Hydrocarbon fraction' [13] [14] [19] [20].

Oxygenated fraction :

TLC examination of the fraction over silica gel G layers using petroleum ether - pyridine (95:5) as solvent system, thymol as reference compound, suitable spray reagents viz. $SbCl_5$:CCl_4 in 1:4 for ethers and fluorescein - bromine for esters for visual identification showed the prescence of methyl eugenol (R_T = 2.3), methyl iso-eugenol (R_T= 2.2) and eugenol acetate (R_T = 2.5). Prescence of these compounds was confirmed by Co-TLC. Identity of the compounds was finally confirmed by PMR/ Mass spectroscopy of the compounds obtained from preparative chromatography of the 'Oxygenated fraction' [13] [14] [19] [20].

GAS CHROMATOGRAPHY OF THE OIL :

Chemical composition of the oil was also studied by Gas chromatography using Perkin Elmer Model 3920. In order to know the difference in chemical composition, the two oil samples viz. from the plants distributed at Rishikesh and from the plants raised at Haldwani, were analysed by Gas chromatography under similar parameters. Different constituents were identified by 'injection method' and comparison of retention time under the same operating parameters using pure compounds. The conditions maintained were :

Column	3.6m long, 2.0 mm I.D.,
Stationary phase	-UCCW 10% in chromosorb, 80-100 mesh.
Column temperature	70-200°C ; @ 2° per minute.
Chart speed	5 mm per minute.
Detector & injector	

temperature	......	250° C.
Attenuation	......	512 & 256.
Range	......	10
Sample size	......	4µl-6µl (1:30 solution in $CHCl_3$).

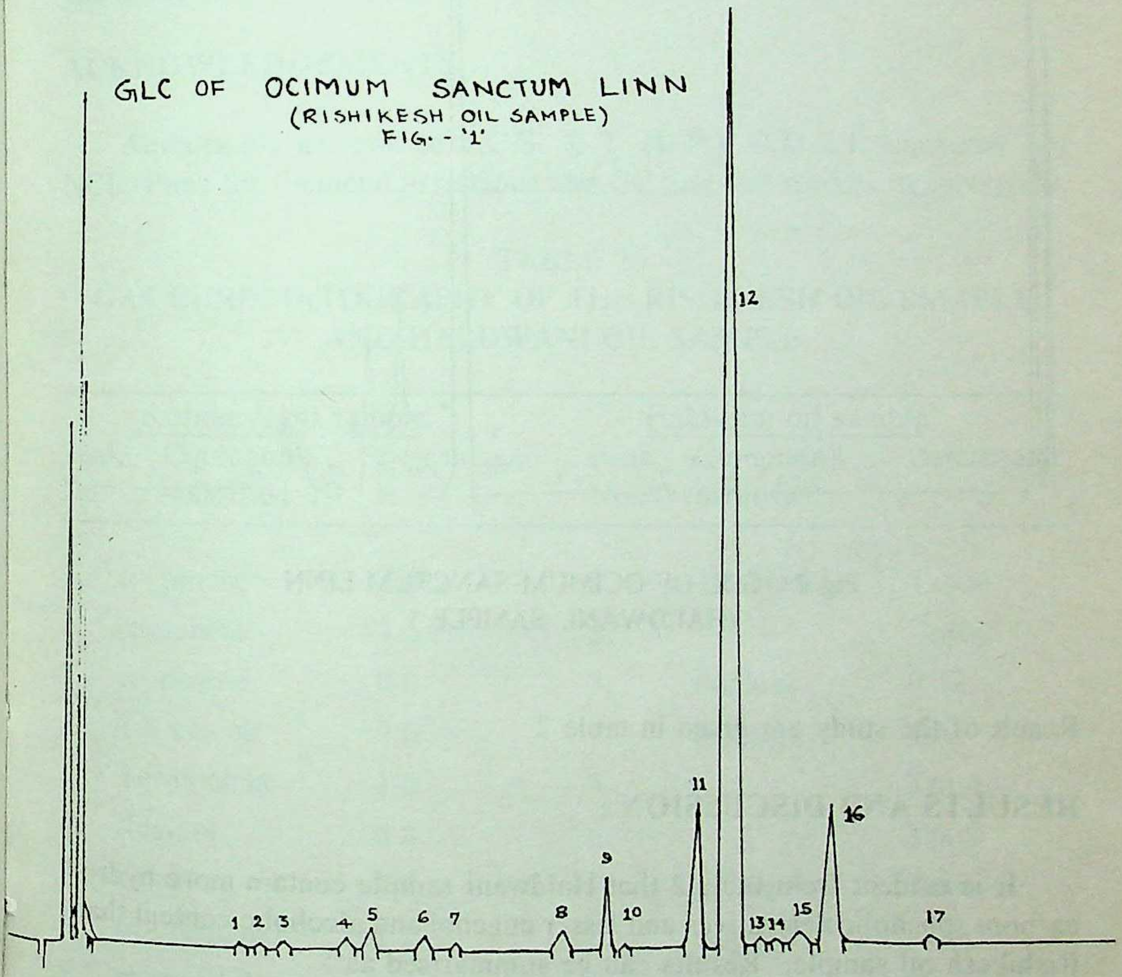

Fig 1 : GLC OF OCIMUM SANCTUM LINN
(RISHIKESH OIL SAMPLE)

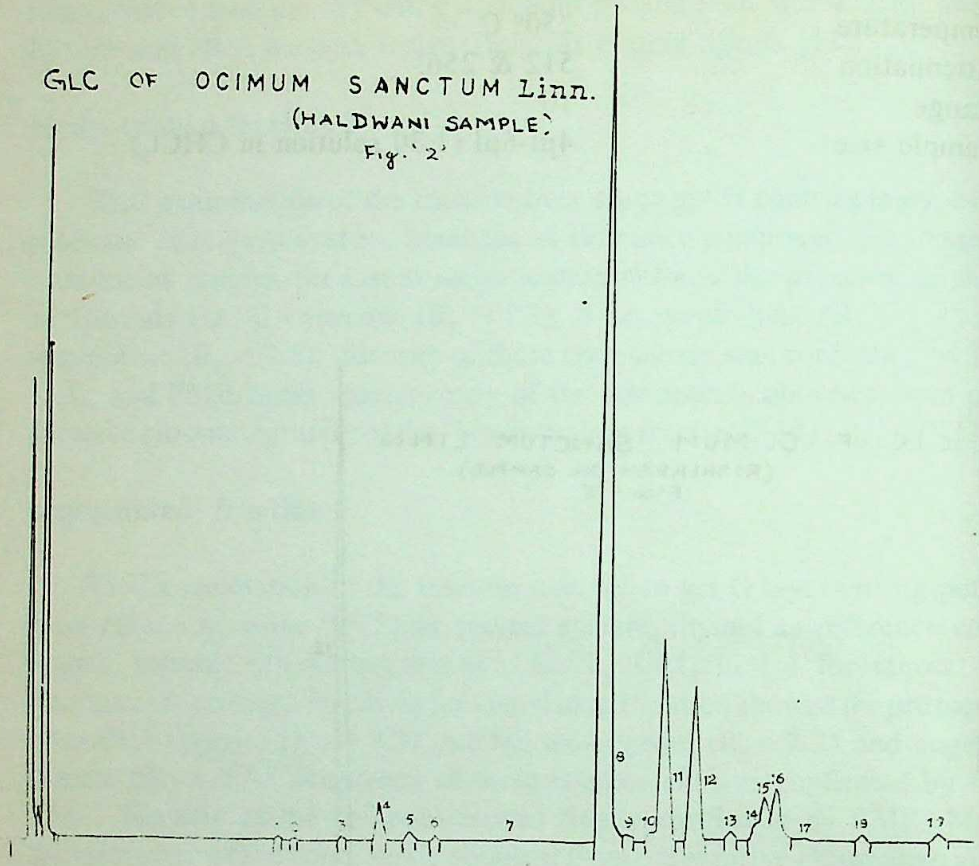

Fig 2 : GLC OF OCIMUM SANCTUM LINN
(HALDWANI SAMPLE)

Result of the study are given in table 2.

RESULTS AND DISCUSSION :

It is evident from table 2 that Haldwani sample contain more hydrocarbons, phenolic derivatives and lesser eugenol and alcoholic content than Rishikesh oil sample. Results can be summarised as :

	Haldwani sample	Rishikesh sample
Hydrocarbons	20.72%	12.85%
Phenolic derivatives	24.00%	0.6%
Eugenol	61.4%	66.5%
Alcoholic content	1.72%	7.3%

Difference in chemical composition of two oil samples may be attributed to the change of locality/place of cultivation of the plant. Further work is needed to ascertain the actual factors responsible for the difference. The authors have identified twenty compounds in the oil of *O. sanctum* as against nine compounds viz. linalool, geraniol, citronellol, cineole, methyl chavicol, eugenol, carveol, methyl eugenol and caryophyllene, reported as a result of work carried out by number of workers mentioned in literature [2] [3] [5] [12] [15]. Further, Rishikesh oil sample contanining 66.5 percent eugenol can serve commercially as a substitute for Oil of cloves.

ACKNOWLEDGEMENTS :

Authors are grateful to S.C.S. & T. (U.P.); C.D.R.I. Lucknow and NCL, Pune for financial assistance and GC and MS studies, respectively.

TABLE 2
GAS CHROMATOGRAPHY OF THE RISHIKESH OIL SMAPLE AND HALDWANI OIL SAMPLE

Rishikesh oil sample			Haldwani oil sample		
Peak No.	Compound identified	Percentage	Peak No.	Compound identified	Percentage
1.	α- pinene	Trace	1.	1:8 cineole	Trace
2.	camphene	Trace	2.	-	Trace
3.	α- pinene	0.6	3.	linalool	0.12
4.	1:8 cineole	0.6	4.	borneol	1.6
5.	Terpinolene	1.0	5.	-	Trace
6.	linalool	0.8	6.	-	Trace
7.	camphor	0.1	7.	geraniol	Trace
8.	α - terpineol	1.2	8.	eugenol	61.4
9.	citronellol	4.9	9.	-	Trace
10.	geraniol	0.4	10.	-	Trace
11.	bornyl acetate	11.9	11.	methyl eugenol	20.15
12.	eugenol	66.5	12.	β-caryophyllene	17.4
13.	-	Trace	13.	humulene	0.68
14.	-	Trace	14.	methyl isoeugenol	1.65

15.	methyl eugenol	0.6	15.	aceto eugenol	2.2
16.	β-caryophyllene	11.1	16.	β-elemene	2.1
17.	β - elemene	0.15	17.	γ-elemene	Trace
18.	cadinene	0.52			
19.	-	0.56			

-denotes in this table the unidentified peaks.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 11-22

PHARMACOGNOSTIC STUDIES IN LEAF OF *LENOTIS NEPETAEFOLIA* L.

S. Jelani* & M. Prabhakar*

(Received 05-06-1991)

ABSTRACT

The paper deals with Pharmacognosy of *Leonotis nepetaefolia* leaf including its morphological, anatomical, chemical constituents and powder analysis. Contrary to the earlier reports the leaves are amphistomatic, stomata are mostly diacytic, few isotricytic. Four types of trichomes viz. unicellular conical hairs, uniseriate conical hairs, uniseriate capitate hairs, uniseriate peltate hairs have been recorded. The venation is Craspido-brochidodromous which is a new pattern. Midrib consists of one large semicircular shaped and petiole with five, one median large arch shaped and four laterals, two on each of the arms of median bundles. Powder analysis microscopically show fragments of epidermis, mesophyll, trichomes, tracheary elements. Positive tests for glycosides, alkaloids polyphenolases, flavones, juglones are recorded.

Keywords : *Leonotis nepetaefolia.* Pharmacognosy.

INTRODUCTION

The leaves of *Leonotis nepetaefolis* are curative and consider to be emmenagogue februfuge, depurative, narcotic, bitter and laxative: used in skin diseses, amenorrhoea and fevers [1,2,16]. The plant is medicinally used by tribe mundas of Chotanagpur. When a mother's breasts are well and milk does not pass through the nipples, the leaf, juices is also expressed and taken with lime juice and run as a febrifuge [9].

MATERIAL AND METHODS

Twigs of *Leonotis nepetaefolia* were collected from plants growing in Osmania University campus, Hyderabad, Andhra Pradesh. The methodology and terms used are after Prabhakar and Ramayya [18]. Leelavathi et al. [11] Prabhakar and Leelavathi [6]

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*Department of Botany, Osmania University, Hyderabad - 500 007, India

Abbreviations used in the text are : Ab: abaxial surface of the organ, Ad: adaxial surface of the organ, D:. diameter, Dist: distribution, E.C.F.: epidermal cell frequency, L/W: length/width, S/F.: stomatal frequency.

OBSERVATIONS AND DISCUSSION

Vernacular names : BENGAL: RAJURCHEI; BOMBAY: Matijer, Matisul; BRAZIL; Cordao de frade; CEYLON: Kasitumpai; GUJARATI:Matizer, Matisul, HINDI: Bejurchei, MARATHI: Dipmal, Ekri, MUNDAR: Agaiajanum, Gharia, Matsuenggelsui, Singgelsui, PORTO RICO: Rascamono, SANTALI: Daredhompo, Jonumdhompo; SIMHALESE; Mahayakwanassa; TELUGU; Rana beri; Hanumantabira; Mulagolimedi.

Morphology: Erect. branched, pubescent, aromatic, herb, upto 2 m high, Root: Tap root Stem: branched, obtusely quandrangular with ridged angles, finely pubescent. Leaves : Simple, opposite-deccusate, ovate 10-13 x 7-9 cm, margin serrate, apex acute, pubescent. Petiole: Four to five cm. filifrm, pubescent. Inflorescence: Axillary, verticillaster, globose, about 6 cm in diameter. Flowers: Small, orange-scarlet, sessile, Bract : linear, 6 mm long strongly spinous, pointed, pubescent, Bracteoles: Two bract-like, 1-1.5 mm. Calyx : Somewhat bilabiate; tube 3 m long; lobes subequal, subulate, 2 mm long.. Corolla: Petals 5, bilipped; tube 9 mm, glabrous below, densely clothed above with scarlet orange hairs; upper lip 11 mm; lower lip 3 lobed 6 mm, lobes oblong. Stamens: Four didynamous, filaments subequal, 4-6 mm, anther 2-celled, cells divaricate, 1.25 mm. Ovary: Truncate, deeply 4-lobbed, pubescent, ovule solitary in each locule on axile placentation. Style: Filiform 1.5 cm glabrous. Stigma: Subequally bilobed. Fruit : Aggregate of 4 nutlets, nutlets oblong, 3-gonous, Flowering and fruiting : October to December.

Microscopic characters

Surface view of leaf : Anticlinally walls of laminar epidermal cells were described to be wavy to sinuate [5]. Presently the epidermal cells are polygonal anisodiametric and rarely polygonal linear. Anticlinal walls are sinuate sinuses being "V" shaped on abaxial but wavy to sinuate on adaxial sides. In none of the Lamiaceae taxa papillate epidermal cells were reported [5,10,15,19], however, presently papillated epidermal cells are observed on both abaxial and abaxial surfaces which is an important diagnostic charcter and has been reported only in few angiospermous taxa. Anticlinal walls are slightly thick and cytoplasmic contents are scanty. Epidermal cells are irregularly arranged and variously oriented Dist : All over the leaf lamina except on veins (Figs. 1&2).

Foliar costal cells and stomatal distribution patterns were considered to be of identification value [1]. The costal cells in the present taxon are mostly polygonal linear and a few rectangular linear. Anticlinal walls are straight and slightly thick. The costal cells are parellely oriented and irregularly arrange. Dist : On all grades of veins.

Type of the stomata in *Leonotis nepetaefolia* was reported to be diacytic [5,10,15,19,20]. However, presently the dominant diacytic stomata are found to be associated with few isotricytic stomata on abaxial side. Stomata are monocyclic. (Fig. 1&2). Shape of the guard cells are elliptic. Size of the stomata on adaxial and abxial surface range from 23-18-15 μm in (length) and 15-12-11 μm (width). Subsidiaries are mostly "a" type rarely of "f" type and are indistinct from epidermal cells (Fig. 1&2). Dist :The leaves were reported to be hypostomatic by Inamdar and Bhatt [5] however presently they are observed to be amphistomatic and distributed all over the leaf lamina except on veins of both the surfaces (Figs. 1&2).

In the past glandular (uni or bicelled head) and non glandularl (uniseriate and multicellular) trichomes were reported [5,10,15,19,20]. However there were no reports on presence of unicellular conical hairs which are presently observed. Besides other types like uniseriate conical hairs, uniseriate capitate hairs, uniseriate peltate hairs are also observed which are described below:

1. Unicellular conical hair : Foot : consisting of the basal end of the hair, indistinct, contents slightly dense; walls thin. Body : short or long, conical, apex acute, contents scanty; walls thin; surface smooth (Fig. 6).

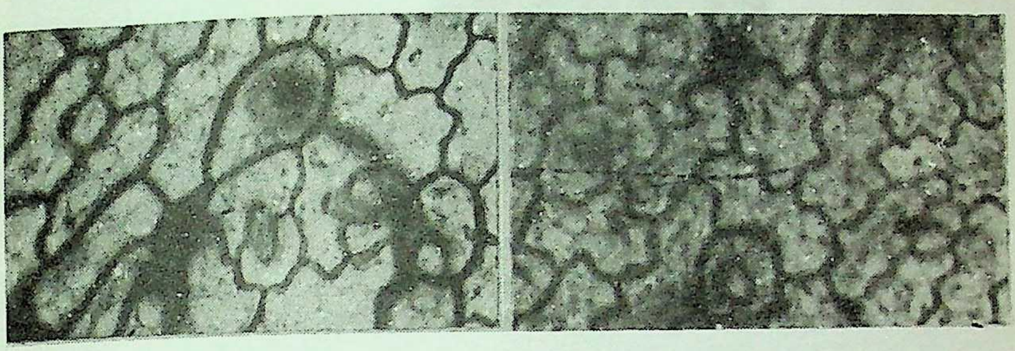
2. Uniseriate conical hairs : Foot : One - celled, embedded or slightly projected above the epidermis, indistinct, contents scanty, walls thin. Body: Conical unicellular to uniseriate, 1 to 3 - celled in length; cells longer than broad contents scanty, walls thin, surface smooth (Fig. 7).

3. Uniseriate capitate hairs : Foot : One-celled, embedded, or slightly projected above the epidermis indistinct, usually broader than long; contents scanty; walls sthin Stalk : one to three celled; two tired; broader than long or squarish, contents scanty; walls thin surface smooth. Head : Eight celled, cells longer than broad; contents dense; walls thin surface smooth (Fig. 8).

4. Uniseriate peltate hairs : Foot : One-celled embedded or slightly projected above the epidermis, indistinct, usually broader than long; contents scanty; walls thin. Stalk : One to three-celled, two tired, broader than long or squarish, contents scanty; walls thin surface smooth, Head : Four or eight celled, cells longer than broad; contents dense; walls thin; surface smooth (Fig. 9).

Foliar venation as taxonomic tool has been since long time in use [16] which is equally important in Pharmacognostic studies [6-8,12]. Presently it is observed that most of the secondaries in *Leonotis* show branching. Further the basal secondaries terminate at margins (like in Craspidodromous pattern) and distal secondaries join the superadjacent secondaries forming loops (as in Brochidodromous pattern). This pattern of venation do not fit under any of the pattern described by Hickey [4] or Melville [14]. Hence presently described as a new pattern Craspido-brochidodromous which can be used as significant character for identification of the foliar drugs. (Fig. 3). Secondaries are six pairs, sub-opposite to opposite to alternate. The proximal loop forming branches joing the super adjacent secondaries at right angles to obtuse angle. Tertiaries are percurrent and their angle of origin is acute: right angle (AR). The relationship of the tertiaries to midvein is oblique to perpendicular and their course is retroflexed and predominantly opposite. Higher order is observed upto 5° vein. Areoles are polygonal, regular in orientation and 600/cm^2. Veinlets are simple and branched once, veinilets are 0-1/areole and 500/cm^2 (Fig. 5).

Sectional view of leaf : In transaction of leaf lamina is ribbed abaxially at primary, secondary and tertiary veins but adaxially inconspicuously ribbed only at primary veins (Fig. 3). Epidermal cells are circular to oval shaped, abaxial ones are smaller in size. (on lamina wings Ad: L/W=41-25-19/19-14-11; Ab:L/W = 30-21-15/15-13-11; midrib D: 19-15-11/16-28-8 μm) with thin cuticle and scanty contents. Guard cells with upper ledges. Palisade one layered. Cells are cylindricl and loosely arranged, (L/W: 79-57-49/23-15-11 μm). Near midvein and major lateral veins on both sides, hypodermis consists of 1-4 layered compactly arranged angular collenchyma (D=27-17-11 μm) and rest the ground tissue is parenchymatous. Parenchyma is 3 to 4 layers; cells polygonal to circular in shape(D-46-33-27 μm). Midvein consists of central large, semi-circular shaped single vascular bundle which is collateral and apericyclic. (Figs. 3). Tracheary elements are polygonal to circular (D=19-13-8 μm) more in midvein and secondary veins; few in others, arranged in rows. Secondary wall thickenings annular (free) helical (helices single or double). Perforation plate is simple.



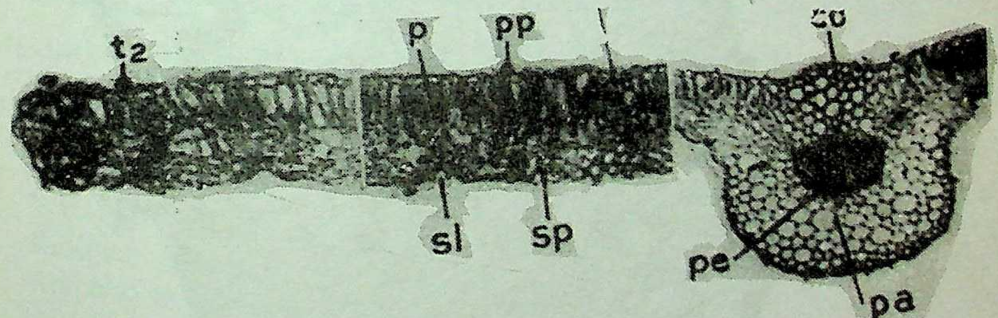

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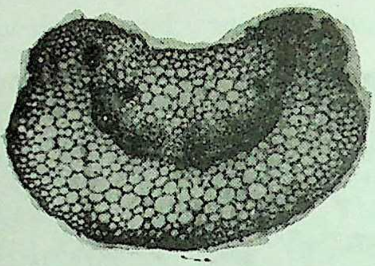
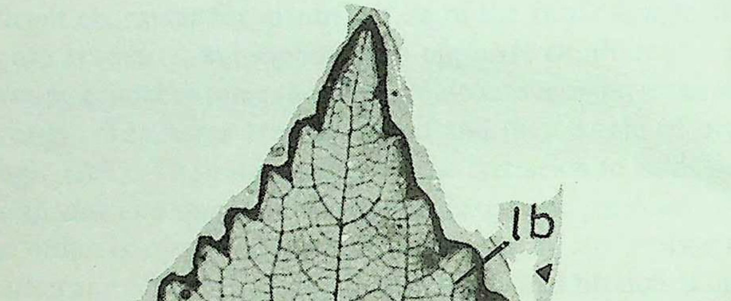
Fig. 1-3





5

Fig. 4 & 5

In transection the petiole is circular and adaxially groved (Fig. 4). Epidermal cells are circular to oval in shape (D=15-13-11 μm). Ground tissue is heterogenous with 2 to 3 layered compaetly arranged collenchymatous hypodermal circular to polygonal.cells (19-17-15 μm) with scanty contents. Rest of the ground tissue is 5 to 8-layered parenchyma. Cells are polygonal (D=87-66-46 μm) with thin walls and scantly contents. Vascular bundles are 5; the median one is large, lunar shaped and the four small laterals, two on each of the arms of median bundle (Fig. 4). Tracheary elements are polygonal to circular (23-17-13 μm) arranged in rows. Secondary wall thickenings are annular (Free), helical (helices single or double) with simple perforation.

Powder Study

The powder is light brown, aromatic. Microscopic examination of the powder revealed the following elements 1) Fragments of epidermis with wavy or sinuate anticlinal walls and diacytic stomata (Fig. 10); 2) Fragments of costae or petiole showing small polygonal epidermal cells without stomata (Fig. 11); 3) Full or bits of uniseriate conical capitate, and peltate hairs (Fig. 12); 4) groups or isolated palisade, spongy and ground parenchyma and collenchyma (Fig. 13-16); 6) bits of vessels with annular, hellical thickeings (Fig. 17).

Besides the microscopic observations the cold extracts of the powder in different solvents and the moist and dry powders are observed under ordinary light and Ultraviolet light to record the colour changes. (Colours were compared with colour index [13] The details are given in table.

OBSERVATION OF LEAF UNDER ORDINARY AND U. V. LIGHT

Sl. No. Powder	Ordinary Light	U. V. light
1. Dried powder	CB	LB
2. Water extract	RM	DG
3. Residue	DP	GB
4. Alcholic extract	GY	GB
5. Alcholic residue	GY	DP
6. Chloroform extract	OB	GB
7. Chloroform residue	BB	OB

BB. Brown black; CB. clay brown; DG. dark green; DP. dull brown; GB. greenish brown; GY. greenish yellow; LB. light brown; OB. olive brown; RM. ripe mango colour.

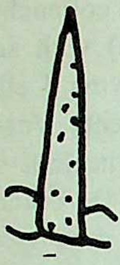


Fig. 6

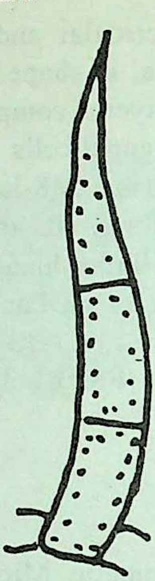


Fig. 7

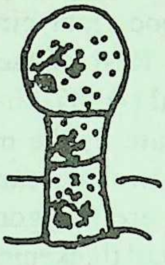


Fig. 8

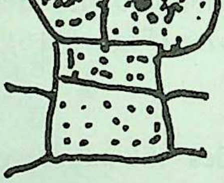


Fig. 9

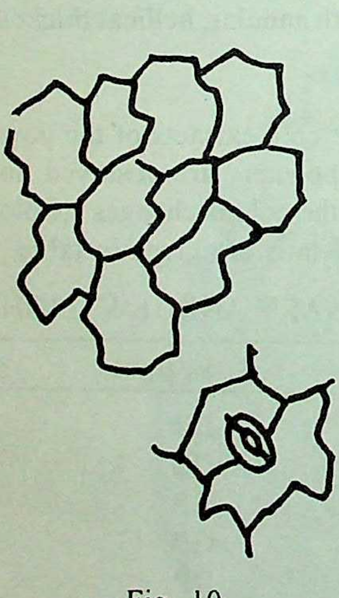


Fig. 10

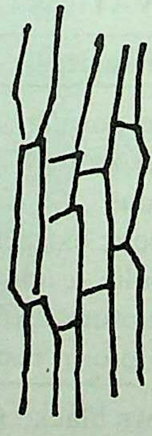


Fig. 11

Fig. 6-11

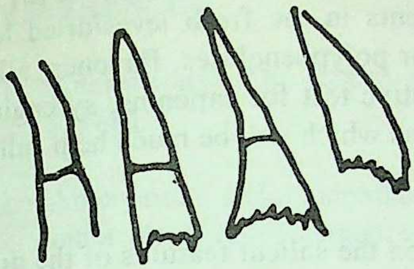
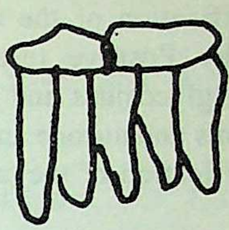


Fig. 12



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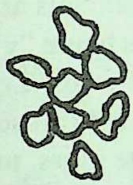

Fig. 14

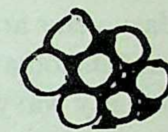

Fig. 15

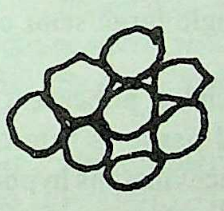

Fig. 16

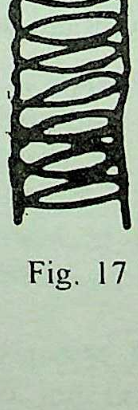

Fig. 17

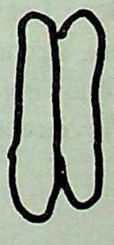

Fig. 18

Fig. 12-17

Following the simle phytochemical tests recommended by Gibbs [3] for identification of the chemical constitutents in the fresh leves/dried leaf powder. Positive tests were recorded for polyphenolases, flavones, alkaloids, glycosides and juglones, and negetive test for saponins, syrengins, acubins and aurone and leucoanthocyanins which will be much help full in identification of the taxa in powder form.

In the light of the present investigation the salient features of the drug which would be useful in identification in its different states are follows :

Leaf : Leaves ovate, base sub-cordate, apex acute; margin serrate, densely hairy, green 10-13 x 7-9 cm with Craspido-brochidodromous venation.

Secondary veins are six pairs, mostly branched. Epidermal cells are mostly polygonal anisodiametric, sides abaxially sinuate, sinuses being "v" shaped and adaxially wavy to sinuate. Costal cells distinct on all veins. Leaves are amphistomatic; stomata mostly diacytic rarely associated with anisotricytic. Unicellular and uniseriate conical hairs uniseriate capitate hairs, uniseriate peltate hairs present all over except on margin. In transection leaf is bifacial, abaxially ribbed at midvein, secondary vein, tertiary vein and feebly at adaxial midvein. Palisade is one layered and spongy mesophyll is 7-8 layered. Midvein with single large semi circular shaped vascular bundle.

Petiole : Five cm long, slender, pubescent, In transection it is circular and adaxially groved. Ground tissue is heterogenous consisting of 3-layered, compactly arranged collenchymatous hypodermis and 5-8 layered parenchymatous tissue. Vascular bundles are five; the median one is large, lunar shaped and the four small laterals two on each of the arms of median bundle.

Leaf powder : Leaf powder is brown, aromatic microscopically fragments of epidermis, stomata, mesophyll, trichomes and tracheary elements can be observed. Following the simle tests [3] positive test for glycosides, polyphenolases, alkaloids flavounes, juglones are recorded.

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EXPLANATON TO FIGURES

Plate 1. *Leonotis nepetaefolia* L. Fig. 1 & 2. Adaxial and abaxial epidermis of leaf lamina (x 440); 3. Transection of leaf lamina from midrib (x 52), lamina (x96) and margin (x 102); 4. Sectional view of petiole (x 75); 5. Venation pattern of leaf lamina. (CO. Collenchyma; 1b. loop forming branch; Pe. Phloem parenchyma; Ph. Phloem. pp. Papilate: Sc. Secretory cell ; Sp. Spongy parenchyma X. xylem; 1°. Primary vein; 2°b. Secondary vein branched. 3°. Tertiary vein).

Figs. 6. Unicellular conical hair 7. Uniseriate conical hair; 8. Uniseriate capitate hair. 9. Uniseriate peltate hair; 10. Pieces of the epidermal cells; 11. costal cells; 12. Full or bits of trichomes; 13 & 18. Palisade parenchyma; 14. Spongy parenchyma; 15. Collenchyma cells; 16. Ground parenchyma; 17. Tracheary elements.

Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 23-30

THERMODYNAMICS AND STOCHASTICS OF BIOLOGICAL GROWTH

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(Received 19-01-1991 and after revision 28-06-91)

ABSTRACT

A non-equilibrium thermodynamic model of biolgical growth together with the stochastic analysis of nonequilibrium fluctuation has been presented.

AMS (MOS) Subject classification (1980) : 92A7

Keywords : Gompertz equation, Entropy production, Quasi-stationary state, Non-equilibrium fluctuation.

INTRODUCTION

The study of growth, development and aging of living organism is an important field of biological sciences. A purely thermodynamic theory of growth and aging is inadequate in view of the stochastic nature due to internal noises irrespective of the random effect of the environment. The non-equilibrium thermodynamic model combined with the stochastic theory of non-equilibrium fluctuation is the proper physico-chemical theory of biological growth and aging [1]. In some earlier papers [2-4] we have tried to develop non-equilibrium thermodynamic and stochastic models of biological growth and aging. In the present paper we shall study the thermodynamic and stochastic behaviour of a growth model which was earlier derived from stochastic thermodynamic model of the system.

BIOLOGICAL GROWTH : THERMODYNAMIC AND STOCHASTIC MODELS

The living system is an open system. For the applicability of the thermodynamics of irreversible processes to the process of biological growth and development, we must have the entropy balance equation for open system.

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$$\frac{1}{m}\frac{dS}{dt}=\frac{1}{m}\frac{d_iS}{dt}+\frac{1}{m}\frac{d_eS}{dt} \tag{1}$$

where S is the entropy of the system at any time t, d_iS is the entropy production due to irreversible processes inside the system and d_eS is the flow of entropy due to the exchange with the environment. A fundamental quantity of non-equilibrium processes is the dissipative function ψ which is related to the rate of entropy production by the relation

$$\psi = T\frac{d_iS}{dt} \tag{2}$$

where T is the temperature of the system. The basic problem of the non-equilibrium thermodynamics of biological growth and development is to find the appropriate dissipative function for the living system. In the earlier paper [2] we have derived the expression of the dissipative function ψ as

$$\psi = A\, e^{-\alpha t} \tag{3}$$

where A is a constant and α is a parameter. Following Zotin and Zotina [1], let us take the specific current weight as the thermodynamic flux

$$\frac{1}{m}\frac{dm}{dt}=J \tag{4}$$

where m is the growth parameter (weight or mass) of the system. If X be the thermodynamic force corresponding to the flux J, then we have [5]

$$\psi = JX = A\, e^{-\alpha t} \tag{5}$$

Using the linear phenomenological relation $J = LX$, we get finally

$$\frac{dm}{dt}=a\, e^{-\beta t} m \tag{6}$$

where $a = (LA)^{1/2}$, and $\beta = \alpha/2$ is the senescence parameter.

The equation (6) is the famous Gompertzian equation which describes a wide variety of growth such as tumour growth, embryonic growth, growth of animals and men [6].

Let us now study the stochastic behaviour of the Gompertz growth process described by the equation (6). For this let us extend the deterministic equation (6) to the form of the stochastic differential equation.

$$\frac{dm}{dt} = a\, e^{-\beta t} m + f(t)$$

$$= A(t)m + f(t) \tag{7}$$

where $A(t) = a\, e^{-\beta t}$ is the time - dependent drift coefficient and $f(t)$ is the random perturbation which is assumed to be a whitenoise of intensity ε satisfying the conditions

$$< f(t) > = 0 \text{ and } < f(t_1)\, f(t_2) > = 2\, \varepsilon\, \delta\, (t_2 - t_1) \tag{8}$$

where < > represents the average over the stochastic process. The equation (7) is generalization of the famous Langevin equation with time - dependent drift coefficient.

The stochastic behaviour of the process described by equation like (7) is manifested in the moment equations or moments, particularly the second - order moment of the growth parameter $m(t)$. The non-stationary stochastic behaviour of the stochastic Gompertzian equation (7) has been studied elsewhere [4]. In this paper we are, however, interested in the quasi-stationary state of the system. Such a state is characterized by the slow variation of the drift coefficient $A(t)$ and the diffusion coefficient ε which is assumed here to be a constant. The stationary (or steady) non-equilibrium state of living systems occur at different times for differtent living organisms. As far example, for human it generally appear about the age 25 to 30. During the steady non-equilibrium state or homeostasis the growth parameter remains practically constant, but at the quasi-stationary and non-stationary states it is subject to fluctuation due to innumerous internal random effects irrespective of the external or environmental influences. The study of the stochastic behaviour or fluctuations of the growth parameter for quasi-stationary state which is very near to the stationary non-equilibrium is of great biological importance [7].

The mean square fluctuation of the growth parameter $m(t)$ satisfying the stochastic differential equation (7) is given by a generalized form of the famous Nyquist's relation [2].

$$< m^2(t) >= \frac{\varepsilon}{\pi}\int_{-\infty}^{\infty}\left|\frac{A(t)}{(A(t)+i\omega)}+\frac{A(t)}{(A(t)+i\omega)^3}\right|^2 d\omega \qquad (9)$$

Using $A(t) = a\, e^{-\beta t}$, $A(t) = -\beta A(t)$ and evaluating the integrals appearing in (9), we have

$$< m^2(t) >= \varepsilon\left[\frac{1}{A(t)}-\frac{\beta}{2A^2(t)}+\frac{3\beta^2}{8A^3(t)}\right] \qquad (10)$$

Let us now use the condition of quasi-stationarity. For the problem under consideration this is satisfied as stated earlier by the criteria of slow - variation of the drift - coefficient $A(t) = a\, e^{-\beta t}$ i.e. by the the condition

$$\left|\frac{A(t)}{A(t)}\right| << 1 \, or \left|\beta\right| << 1 \qquad (11)$$

Neglecting terms containing β and its higher powers, we have

$$< m^2(t) >= \frac{\varepsilon}{A(t)} = \left(\frac{\varepsilon}{a}\right) exp\ (\beta t) \qquad (12)$$

showing that the fluctuation of the growth parameter (mass or weight) increases exponentially with time unless there is some restriction on the order of (β t). Apart from the fact the fluctuation becomnes infinitely large, there is another difficulty with the Gompertzian model equation (6). The equation (6) shows that the stationary or steady growth of the living system is reached after an infinitely large time. This is practically impossible for living organisms having terminated growth and finite life span. All these show that there must be some restriction on the order of the exponent (βt). In the next section we are going to resolve all these difficulties faced by Gompertzian Model.

A LINEAR GROWTH MODEL: THERMODYNAMICS AND STOCHASTICS

To resolve the difficulties faced by the Gompertz equation (6), we assume that

$$|\beta t| << \qquad (13)$$

Under this condition, the equation (6) is approximated to

$$\frac{1}{m}\frac{dm}{dt} = J = a\beta\left[\frac{1}{\beta} - t\right] = L_0(T_m - t) \qquad (14)$$

where $1/\beta = T_m$ is the maximum age, $L_0 = a\beta$ is the phenomenological or kinetic coefficient. If we take $= T_m - t$ as proportional to the thermodynamic force corresponding to the flux J, the relation (14) becomes the linear phenomenological relation between the thermodynamic flux and force. The linear approximation of (6) neglecting quadratic and higher powers of (β t) is justified by two facts. The first is that the linear approximation makes the equation amicable to the theory of linear thermodynamics of irreversible processes and the secondly it removes the infinite time to reach the steady state of the growth and also the infinitely increasing of fluctuation as we shall see a bit latter. Above all the linear model (4) is able in explaining a wide variety of biological phenomena related to growth and development of living organism [1]. It is infact, the starting or basic equation in an early development of non-equilibrium thermodynamics of biological growth by Zotin and Zotina [1]. In this paper we have arrived at it from a more general setting of thermodynamic model of growth of living organism. In terms of the growth parameter m, the equation (14) can be written

$$\frac{dm}{dt} = a\beta(T_m - t)m \qquad (15)$$

Integrating (15) and using the initial conditions

$$t = 0,\ T_m = 0,\ m = m_0 \text{ (say)} \qquad (16)$$

we have

$$(T_m - t)^2 = \frac{2}{a\beta} log\ (m_0 / m) \qquad (17)$$

expressing the thermodynamic force in terms of the growth parameter (mass or weight) of the system.

The rate of entropy production then become

$$\frac{d_iS}{dt} = \frac{1}{T}JX = \frac{a\beta}{T}(T_m - t)^2 \tag{18}$$

Integrating, we have

$$\Delta_i S = (S_i)_f - (S_i)_0 = \frac{a\beta}{T}\int_0^t (T_m - t)^2\, dt \tag{19}$$

Using (17) and (18), the entropy production from the initial state O at time O to the final state f at time t can be expressed as

$$\begin{aligned} \Delta_i S &= \frac{2}{T}\int_{m_0}^{m}(\log m_0 - \log m)\,dm \\ &= \frac{2}{T}\left[(m - m_0) - m \log (m / m_0)\right] \end{aligned} \tag{20}$$

which expresses the entropy production in terms of mass or weight of the system, the temperature T being assumed to be constant during the time $(0,t)$. The expression (20) of entropy production has important sigificance. Its mathematical derivation including its biological implication will be studied later on.

Let us now consider the stochastic behaviour of the linear equation (15). The stochastic extension of (15) is given as before by

$$\frac{dm}{dt} = a\beta(T_m - t)m + f(t) \tag{21}$$

Using either directly the formula (9) with $A(t) = a\beta(t_m - t)$ or the relation (12) with the restriction (13) (i.e.$|\beta t| << 1$) the second - order moment of $m(t)$ is given by

$$< m^2(t) > = \frac{\varepsilon}{a} \tag{22}$$

Thus, in contrast to the case of Gompertz model, the fluctuation does not show infinitely diverging character. The above fluctuation, of course, depends on the intensity ε of the noise and the growth rate α.

CONCLUSION

The paper dealing with the thermodynamic and stochastic models of biological growth can be divided into two parts discussed in section (2) and (3) respectively. The first which deals with the thermodynamic derivation of Gompertz growth and its stochastic behaviour is faced with some difficulties as to the infinite time of reaching steady - state and diverging fluctuation. These difficulties have been resolved in the second part (section (3)) which deals with the linear model derived as a linear approximation of Gompertz equation. This linear model was earlier proposed by Zotin and Zotina [1], here we have arrived at it from a general non-equilibrium setting and studied its stochastic behaviour and stressed its importance in relation to the Gompertzian model of growth of living organism. The whole analysis based on simple mathematical equations or models, there is nothing new numerical or graphical works to be added.

ACKNOWLEDGEMENT

The work was done under a project sanctioned by C.S.I.R. (India). The second author wishes to thank C.S.I.R. for a Junior Research Fellowship. The authors wish to thank the referee whose comments helped the remodelling of the paper.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp. 31-46

ON NONLINEAR STOCHASTIC INTEGRODIFFERENTIAL EQUATIONS INVOLVING ITÔ-CLIFFORD INTEGRALS

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(Received 04.08.1991)

ABSTRACT

The aim of this paper is to study a general class of nonlinear stochastic integrodifferential equations involving Itô-Clifford integrals. We study the problems of existence, uniqueness and stability of the solutions by using the method of successive approximation.

AMS Subject Classification (1991): 60 H 20

Keywords and Phrases: Stochastic Integrodifferential Equations, Itô-Clifford Integrals, Existence and Uniqueness, Stability, successive approximation.

INTRODUCTION

In a recent paper [1] Barnett, Streater and Wilde have introduced the concept of Itô-Clifford integral and established some basic properties of this integral. They have presented in [2] an interesting study relating to existence, uniqueness and stability properties of the solutions of a stochastic differential equation involving Itô-Clifford integral. This newly formulated stochastic integral and corresponding stochastic equations are yet in their developing stage and awaits for further attention by workers in this field. The aim of this paper is to study the properties of the solutions of a general stochastic integrodifferential equation

$$\begin{aligned}(1.1)\qquad dx(t) &= F(t,x(t), A_1x(t), A_2x(t))dt + G(t,x(t), B_1x(t), B_2x(t))\,dw(t)\\ &\quad + dw(t)\,H(t,x(t), P_1x(t), P_2x(t))\end{aligned}$$

where

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$$A_1 x(t) = \int_{t_0}^{t} f_1(t, s, x(s))\, ds, \; A_2 x(t) = \int_{t_0}^{t} f_2(t, s, x(s))\, dw(s),$$

$$B_1 x(t) = \int_{t_0}^{t} g_1(t, s, x(s))\, ds, \; B_2 x(t) = \int_{t_0}^{t} g_2(t, s, x(s))\, dw(s),$$

$$P_1 x(t) = \int_{t_0}^{t} h_1(t, s, x(s))\, ds, \; P_2 x(t) = \int_{t_0}^{t} h_2(t, s, x(s))\, dw(s),$$

and $w(t)$ is Fermion martingale, with initial condition $x(t_0) = x_0$, $t \geq t_0$ and $x_0 \varepsilon L^2(\ell t_o)$.

As in [2] we consider the case of $L^2(R_+)$, $R_+ = (0, \infty)$ with complex conjugation. Let ℓ denote the w^*-algebra generated by the bounded operators $\{ \Psi(z) = C(z) + A(z), z \varepsilon L^2(R_+)\}$ where $C(.), A(.)$ are the Fock fermion creation and annihilation operators respectively and $A(z)$ is the adjoint of $C(z)$; $A(z) = C(z)^*$. For given $t \geq 0$ denote by ℓt the W^* - subalgebra of ℓ generated by the fields $\psi(z)$ for $z \varepsilon L^2(R_+)$ with ess supp $z \subseteq [0, t]$ and the completion of ℓ_t with respect to $\| \cdot \|_p$ by $L^p(\ell_t)$. Therefore, for $1 \leq p < \infty$, $L^p(\ell_t)$ may be considered as a subspace of $L^p(\ell)$. A map $x: R_+ \to L^2(\ell)$ is said to be adapted if $x(t) \varepsilon L^2(\ell_t)$ for each $t \varepsilon R_+$ and a map $Z: (R_+)^2 x L^2(\ell_t) \to L^2(\ell_t)$ is said to be adapted if for any $s \leq t$, $s, t \in R_+$ and $u \in L^2(\ell_t)$ we have $Z(t, s, u) \in L^2(\ell_t)$. Similarly a map $Y: R_+ x [L^2(\ell_t)]^3 \to L^2(\ell_t)$ is said to be adapted if for any $t \in R_+, u_i \varepsilon L^2(\ell_t), i = 1, 2, 3$, we have $Y(t, u_1, u_2 u_3) \in L^2(\ell_t)$. In particular if $x_i : R_+ \to L^2(\ell), i = 1,2,3, z : (R_+)^2 x L^2(\ell) \to L^2(\ell)$ and $Y : R_+ x[L^2(\ell)]^3 \to L^2(\ell)$ are adapted then so are the maps $t \to Z(t, s, x_i(s))$ and $t \to Y(t, x_1(t), x_2(t), x_3(t))$. Let $V_1, \ldots, V_n \varepsilon L^2_{10c}(R_+)$ so that for any $t \geq 0$, $v_i \chi_{[0,t]} \varepsilon L^2(R_+)$, $1 \leq i \leq n$. The Wick-ordered monomial : $\psi(v_1 \chi_{[0,t]}) \cdots \psi(v_n \chi_{[0,t]})$: is denoted by $w(t)$. It is proved in [1., Theorem 2.2, p. 176] that $w(t)$ is an L^∞-martingale adapted to the family $\{\ell_t : t \varepsilon R_+\}$. For detailed notations and definitions see [1,2].

The general formulation of equation (1.1), which in turn contains as a special case the stochastic differential equation studied by Barnett, Streater and Wilde [2], is motivated by the study of general forms of Itô -type stochastic equations by Berger and Mizel [3], Kannan and Bharucha Reid [4] and Pachpatte [5]. In section 2 we formulate our results on the existence, uniqueness and stability of the solutions of equation (1.1) and its more general form. Section 3 deals with the proofs of our main results. The method of successive approximation is used to establish our results.

STATEMENT OF RESULTS

Before stating the main results in this section we list the necessary preliminaries from [1,2] and the hypotheses needed in our subsequent discussion.

Fix $v_1, \dots, v_n \, \varepsilon \, L^{\infty}_{loc}(R_+)$ and suppose that $W(t) = : \psi(v_1 \chi_{[0,t]}) \dots \psi(v_n \chi_{[0,t]}):$ for $t \, \varepsilon \, R_+$. Then $W(s)\, W^*(s) = a(s)\mathbb{1}$, where

$$(2.1) \quad a(s) = \int_0^s \dots \int_0^s |v(s_1, \dots, s_n)|^2 \, ds_1 \dots ds_n,$$

with $v(s_1, \dots, s_n) = (n!)^{1/2} \mathcal{A}(v_n \otimes \dots \otimes v_1)(s_1, \dots, s_n)$ see [1, p. 194]. Let μ denote the measure on R_+ given by the positive increasing function $s \to a(s)$ that is $\mu([\alpha, \beta]) = a(\beta) - a(\alpha)$ for $\alpha \leq \beta$ in R_+ and let $f : R_+ \to L^2(\ell)$ be measurable and adapted with $\int_0^t \|f(s)\|_2^2 \, d\mu(s) < \infty$. It is shown in [1] that the Itô-Clifford stochastic integral satisfies the following property,

$$(2.2) \quad \left\| \int_0^t f(s)\, dW(s) \right\|_2^2 = \int_0^t \|f(s)\|_2^2 \, d\mu(s).$$

Furthermore $\int_0^t dW(s) f(s) = \left(\int_0^t f^*(s)\, dW^*(s)\right)^*$ and $W^*(s)\, W(s) = a(s)\, \mathbb{1}$, therefore we also have

$$(2.3) \quad \left\| \int_0^t dW(s)\, f(s) \right\|_2^2 = \int_0^t \|f(s)\|_2^2 \, d\mu(s),$$

see [1,2].

The equation (1.1) is equivalent to the stochastic integral equation

$$(2.4) \quad x(t) = x_0 + \int_0^t F(s, x(s), A_1 x(s), A_2 x(s))\, ds$$

$$+ \int_0^t G(s, x(s), B_1 x(s), B_2 x(s))\, dW(s) + \int_0^t dW(s)\, H(s, x(s), P_1 x(s), P_2 x(s))$$

The integrals $A_1 x(s)$, $B_1 x(s)$, $P_1 x(s)$ are $L^2(\ell)$ Bochner integrals and the integrals $A_2 x(s)$, $B_2 x(s)$, $P_2 x(s)$ are Itô-Clifford integrals, see [2]. Further, the first integral on right in (2.4) is an $L^2(\ell)$ Bochner integral while the second and third integrals are Itô-Clifford integrals. In order these integrals to be well defined, it is sufficient for the maps $s \to f_1(s, \tau, x(\tau))$, $s \to g_1(s, \tau, x(\tau))$, $s \to h_1(s, \tau, x(\tau))$ and $s \to F(s, x(s), A_1 x(s), A_2 x(s))$, $\tau \leq s$ to be L^2 - continuous and the maps $s \to f_2(s, \tau, x(\tau))$, $s \to g_2(s, \tau, x(\tau))$, $s \to h_2(s, \tau, x(\tau))$, $s \to G(s, x(s), B_1 x(s), B_2 x(s))$ and $s \to H(s, x(s), P_1 x(s), P_2 x(s))$, $\tau \leq s$ are to be adapted and L^2 - continuous. This is the situation when $s \to x(s)$ and f_2, g_2, h_2, G and H are continuous and adapted and f_1, g_1, h_1 and F are contiuous. In order to ensure that the map $s \to x(s)$ is adapted, we assume that f_1, g_1, h_1 and F are adapted. We make use of the above assumption throughout the paper without further mention.

For convenience we list the following assumption :

(H_1) For $x, y, x_k, y_k, z_k \in L^2(\ell)$, $k = 1, 2$ and $t \in [0, T]$ there exist constants K_i, M_i, N_i, $i=1,2$ and K, M, N such that

$$\| f_i(t, s, x) - f_i(t, s, y) \|_2 \leq K_i \| x-y \|_2,$$

$$\| g_i(t, s, x) - g_i(t, s, y) \|_2 \leq M_i \| x-y \|_2,$$

$$\| h_i(t, s, x) - h_i(t, s, y) \|_2 \leq N_i \| x-y \|_2,$$

for all $0 \leq s \leq t \leq T$ and

$$\|F(t, x_1, y_1, z_1) - F(t, x_2, y_2, z_2)\|_2 \leq K(\| x_1 - x_2 \|_2 + \| y_1 - y_2 \|_2 + \| z_1 - z_2 \|_2)$$

$$\|G(t, x_1, y_1, z_1) - G(t, x_2, y_2, z_2)\|_2 \leq M(\| x_1 - x_2 \|_2 + \| y_1 - y_2 \|_2 + \| z_1 - z_2 \|_2)$$

$$\|H(t, x_1, y_1, z_1) - H(t, x_2, y_2, z_2)\|_2 \leq N(\| x_1 - x_2 \|_2 + \| y_1 - y_2 \|_2 + \| z_1 - z_2 \|_2)$$

for all $0 \leq t \leq T$.

We now state our main result on the existence and uniqueness of the solution of equation (1.1)

Theorem 1.: Let the hypothesis (H_1) be fulfilled. Then for any $x_0 \varepsilon L^2 (\ell t_0)$ there exists exactly one L^2-continuous adapted solution of equation (1.1) on $I = [t_0, \infty)$ with $x(t_0) = x_0$.

The result established in Theorem 1 can be extended to the following general form of stochastic integrodifferential equation involving Itô-Clifford integrals driven by $W^{(j)}(t)$, $1 \leq j \leq r$, of the form

$$(2.5)\ dx(t) = \sum_{j=1}^{r} F^{(j)}(t, x(t), A_1^{(j)} x(t), A_2^{(j)} x(t))\, dt$$

$$+ \sum_{j=1}^{r} G^{(j)}(t, x(t), B_1^{(j)} x(t), B_2^{(j)} x(t))\, dw^{(j)}(t)$$

$$+ \sum_{j=1}^{r} dw^{(j)}(t) H(t, x(t), P_1^{(j)} x(t), P_2^{(j)} x(t))$$

for $t \in T$ with initial condition $x(t_0) = x_0 \varepsilon L^2(\ell_{t0})$
where

$$A_1^{(j)} x(t) = \int_{t_0}^{t} f_1^{(j)}(t, s, x(s))\, ds,\quad A_2^{(j)} x(t) = \int_{t_0}^{t} f_2^{(j)}(t, s, x(s))\, dw^{(j)}(s),$$

$$B_1^{(j)} x(t) = \int_{t_0}^{t} g_1^{(j)}(t, s, x(s))\, ds,\quad B_2^{(j)} x(t) = \int_{t_0}^{t} g_2^{(j)}(t, s, x(s))\, dw^{(j)}(s),$$

$$P_1^{(j)} x(t) = \int_{t_0}^{t} h_1^{(j)}(t, s, x(s))\, ds,\quad P_2^{(j)} x(t) = \int_{t_0}^{t} h_2^{(j)}(t, s, x(s))\, dw^{(j)}(s),$$

and $W^{(j)}(t)$ denotes the Wick-ordered monomial

$:\psi(v_1^{(j)} \chi^{(j)}_{[0,t]}) \ldots \psi(v_n^{(j)} \chi^{(j)}_{[0,t]}):$ for $t \in R_+$. We note that $(W^{(j)}(s))(W^{(j)}(s))^* = a^{(j)}(s)\mathbb{1}$ where $a^{(j)}(s)$ is defined as in (2.1) with corresponding changes. Let $\mu^{(j)}$, $1 \leq j \leq r$ denote the measure on R_+ given by the positive increasing function

$s \to a^{(j)}(s)$. We assume that the functions $f_i^{(j)}, g_i^{(j)}, h_i^{(j)}, i = 1,2, j=1,2, \ldots, r$ defined on $(R_+)^2 x L^2(\ell)$ into $L^2(\ell)$ and functions $F^{(j)}, G^{(j)}, H^{(j)}$ defined on $R_+ x[L^2(\ell)]^3$ into $L^2(\ell)$ for $j = 1,2 \ldots r$ and L^2-continuous and adapted.

We make the following assumptions on the functions involved in (2.5). (H_2) for $x, y, x_k, y_k, z_k \in L^2(\ell), k = 1,2$ and $t \in [0,T]$ there exist constants $K_i^{(j)}$, $M_i^{(j)}$, $N_i^{(j)}$ and $K^{(j)}, M^{(j)}, N^{(j)}, i=1, 2, j=1,2\ldots,r$ such that

$$\| f_i^{(j)}(t, s, x) - f_i^{(j)}(t, s, y) \|_2 \leq K_i^{(j)} \| x-y \|_2,$$

$$\| g_i^{(j)}(t, s, x) - g_i^{(j)}(t, s, y) \|_2 \leq M_i^{(j)} \| x-y \|_2,$$

$$\| h_i^{(j)}(t, s, x) - h_i^{(j)}(t, s, y) \|_2 \leq N_i^{(j)} \| x-y \|_2,$$

for all $0 \leq s \leq t \leq T$ and

$$\|F^{(j)}(t, x_1, y_1, z_1) - F^{(j)}(t, x_2, y_2, z_2)\|_2 \leq K^{(j)}(\|x_1 - x_2\|_2 + \|y_1 - y_2\|_2 + \|z_1 - z_2\|_2)$$

$$\|G^{(j)}(t, x_1, y_1, z_1) - G^{(j)}(t, x_2, y_2, z_2)\|_2 \leq M^{(j)}(\|x_1 - x_2\|_2 + \|y_1 - y_2\|_2 + \|z_1 - z_2\|_2)$$

$$\|H^{(j)}(t, x_1, y_1, z_1) - H^{(j)}(t, x_2, y_2, z_2)\|_2 \leq N^{(j)}(\|x_1 - x_2\|_2 + \|y_1 - y_2\|_2 + \|z_1 - z_2\|_2)$$

for all $0 \leq t \leq T$.

Our next theorem gives sufficient conditions for existence and uniqueness of the solution of equation (2.5).

Theorem 2. : Suppose the hypothesis (H_2) holds. Then for any $x_0 \varepsilon L^2(\ell_{t_0})$ there exists a unique L^2- continuous adapted solution of the equation (2.5) on I with $x(t_0) = x_0$.

Remark : We note that Theorems 2.1 and 2.2 established in [2] are special cases of our Theorems 1 and 2 respectively. An interesting study relating to more general Itô type stochastic integrodifferential equations can be found in recent papers by Berger and Mizel [3] and Pachpatte [5].

The following theorems deal with the stability of the solutions of equations (1.1) and (2.5).

Theorem 3.: Assume that the hypothesis (H_1) holds. Let $y(t)$ and $z(t)$ for

$\geq t_o$ be the solutions of equation (1.1) with initial conditions $y(t_0) = x_0$ and $z(t_o) = x_o^*$ respectively with $x_0, x_0^* \in L^2(\ell_{t0})$. For given $T > t_0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x_0 - x^*_0\|_2 < \delta$ then $\| y(t) - z(t) \|_2 < \varepsilon$ for all $t \in [t_0, T]$.

Theorem 4.: Let the hypothesis (H_2) hold. Suppose $y(t)$ and $z(t)$ for $t \geq t_0$ are the solutions of the stochastic integrodifferential equation (2.5) with initial conditions $y(t_0) = x_0$ and $z(t_0) = x_0^*$ respectively and $x_0, x_0^* \in L^2(\ell_{t0})$. For given $T > t_0$ and $\in > 0$ there exists $\delta > 0$ such that if $\| x_0 - x_0^* \|_2 < \delta$ then $\| y(t) - z(t) \|_2 < \in$ for all $t \in [t_0, T]$.

Proofs of Theorems 1 and 2

Let T be fixed in I such that $t_0 \leq t \leq T$. We define

$$(3.1) \quad x^{(0)}(t) = x_0,$$

and for $n \geq 0$,

$$(3.2)\ x^{(n+1)}(t) = x_0 + \int_{t_0}^{t} F(s, x^{(n)}(s), A_1 x^{(n)}(s), A_2 x^{(n)}(s))\, ds$$

$$+ \int_{t_0}^{t} G(s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s))\, dw(s)$$

$$+ \int_{t_0}^{t} dw(s) H(s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s)).$$

We first show that the process $x^{(n)}(t)$, $n \geq 1$ is well defined and it is adapted L^2- continuous on $[t_0, T]$. The functions $f_i\ (s, \tau, x_0)$, $g_i(s, \tau, x_0)$, $h_i(s, \tau, x_0)$, $i = 1, 2, F\ (s, x_0 A_1 x^{(0)}(s), A_2\, x^{(0)}(s))$, $G(s, x_0\ B_1 x^{(0)}(s), B_2 x^{(0)}(s))$ and $H(s, x_0, P_1 x^{(0)}(s), .P_2 x^{(0)}(s))$ are adapted and L^2-continuous for $t_0 \leq \tau \leq s \leq T$. Thus it is easy to see that $x^{(1)}(t)$ is well defined for $t_0 \leq \tau \leq s \leq T$. Also we observe from the fact that continuity of a function implies boundedness of the function on compact intervals and using the isometry property we see that $t \rightarrow x^{(1)}$ is continuous map:

$$[t_0 T] \rightarrow L^2(\ell).$$

Assume that $x^{(n)}(t)$ is adapted and L^2-continuous on $[t_0, T]$. Then, each of the functions $f_i(s, \tau, x^{(n)}(\tau))$, $g_i(s, \tau, x^{(n)}(\tau))$, $h_i(s, \tau, x^{(n)}(\tau))$, $i = 1, 2$, $F(s, x^{(n)}(s), A_1 x^{(n)}(s), A_2 x^{(n)}(s))$, $G(s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s))$ and $H(s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s))$ is adapted and L^2-continuous on $[t_0, T]$ and hence bounded. It can be easily seen that $x^{(n+1)}(t)$ is adapted and by using isometrics (2.2) and (2.3) we observe that $t \to x^{(n+1)}(t)$ is L^2-continuous on $[t_0, T]$. This proves that the process $x^{(n)}(t)$, $n \geq 1$ is well defined, adapted and L^2-continuous on $[t_o, T]$

We next consider the convergence of the sequence $\{x_n(t)\}$, $n \geq 0$ defined by (3.1)-(3.2). From (3.2) and the fact that $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ we observe that

$$
\begin{aligned}
(3.3) \quad \| x^{(n+1)}(t) - x^{(n)}(t) \|_2^2 \leq\ & 3 \Big\| \int_{t_0}^{t} [F(s, x^{(n)}(s), A_1 x^{(n)}(s), A_2 x^{(n)}(s)) \\
& - F(s, x^{(n-1)}(s), A_1 x^{(n-1)}(s), A_2 x^{(n-1)}(s))]\, ds \Big\|_2^2 \\
& + 3 \Big\| \int_{t_0}^{t} [G(s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s)) \\
& - G(s, x^{(n-1)}(s), B_1 x^{(n-1)}(s), B_2 x^{(n-1)}(s))]\, dw(s) \Big\|_2^2 \\
& + 3 \Big\| \int_{t_0}^{t} dw(s) [H(s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s)) \\
& - H(s, x^{(n-1)}(s), P_1 x^{(n-1)}(s), P_2 x^{(n-1)}(s))] \Big\|_2^2 .
\end{aligned}
$$

From the isometries (2.2) and (2.3) the boundedness of the wave functions v_1, ..., v_n over $[t_0, T]$, the hyposheris (H_1), $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ and Schwarz inequality we observe that

$$
\begin{aligned}
(3.4) \quad & \Big\| \int_{t_0}^{t} [F(s, x^{(n)}(s), A_1 x^{(n)}(s), A_2 x^{(n)}(s)) \\
& - F(s, x^{(n-1)}(s), A_1 x^{(n-1)}(s), A_2 x^{(n-1)}(s))]\, ds \Big\|_2^2
\end{aligned}
$$

$$\leq K^2 (t - t_0) \int_{t_0}^{t} [\| x^{(n)}(s) - x^{(n-1)}(s)\|_2$$

$$+ K_1 \int_{t_0}^{s} \| x^{(n)}(\tau) - x^{(n-1)}(\tau)\|_2 \, d\tau$$

$$+ (K_2^2 Q_1^2 \int_{t_0}^{s} \|x^{(n)}(\tau) - x^{(n-1)}(\tau)\|_2^2 \, d\tau)^{1/2}]^2 \, ds$$

$$\leq 3K^2(T - t_0)\, Q_1^* \int_{t_0}^{t} \| x^{(n)}(s) - x^{(n-1)}(s)\|_2^2 \, ds,$$

(3.5)
$$\| \int_{t_0}^{t} [G (s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s))$$

$$- G (s, x^{(n-1)}(s), B_1 x^{(n-1)}(s), B_2 x^{(n-1)}(s))] \, dw(s)\|_2^2$$

$$= \int_{t_0}^{t} \| [G (s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s))$$

$$- G (s, x^{(n-1)}(s), B_1 x^{(n-1)}(s), B_2 x^{(n-1)}(s))] \|_2^2 \, d\mu(s)$$

$$\leq C_1^2 \int_{t_0}^{t} \|G (s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s))$$

$$- G (s, x^{(n-1)}(s), B_1 x^{(n-1)}(s), B_2 x^{(n-1)}(s)) \|_2^2 \, ds$$

$$\leq 3C_1^2 M^2 Q_2^* \int_{t_0}^{t} \|x^{(n)}(s) - x^{(n-1)}(s) \|_2^2 \, ds,$$

and

(3.6)
$$\| \int_{t_0}^{t} dW(s) \, [H (s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s))$$

$$- H(s, x^{(n-1)}(s), P_1 x^{(n-1)}(s), P_2 x^{(n-1)}(s))]||_2^2$$

$$= \int_{t_0}^{t} \| H(s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s))$$

$$- H(s, x^{(n-1)}(s), P_1 x^{(n-1)}(s), P_2 x^{(n-1)}(s)) \|_2^2 d\mu(s)$$

$$\leq C_2^2 \int_{t_0}^{t} \|H(s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s))$$

$$- H(s, x^{(n-1)}(s), P_1 x^{(n-1)}(s), P_2 x^{(n-1)}(s)) \|_2^2 ds$$

$$\leq 3C_2^2 N^2 Q_3^* \int_{t_0}^{t} \|x^{(n)}(s) - x^{(n-1)}(s)\|_2^2 ds,$$

where

$$Q_1^* = \left[1 + \frac{K_1^2 (T - t_0)^2}{2} + Q_1^2 K_2^2 (T - t_0)\right]$$

$$Q_2^* = \left[1 + \frac{M_1^2 (T - t_0)^2}{2} + Q_2^2 M_2^2 (T - t_0)\right]$$

$$Q_3^* = \left[1 + \frac{N_1^2 (T - t_0)^2}{2} + Q_3^2 N_2^2 (T - t_0)\right]$$

and C_1, C_2, Q_1, Q_2 and Q_3 are constants depending on t_0 and T. Using (3.4) - (3.6) in (3.3) we get

$$(3.7) \quad \| x^{(n+1)}(t) - x^{(n)}(t) \|_2^2 \leq D \int_{t_0}^{t} \| x^{(n)}(s) - x^{(n-1)}(s) \|_2^2 ds,$$

where

$$(3.8) \quad D = 9\,[\,K^2 (T - t_0)\, Q_1^* + C_1^2 M^2 Q_2^* + C_2^2 N^2 Q_3^*].$$

By succesive iterations on (3.7) we have

$$\| x^{(n+1)}(t) - x^{(n)}(t)\|_2^2$$

$$\leq D^n \int_{t_0}^{t} \int_{t_0}^{s_{n-1}} \cdots \int_{t_0}^{s_1} \| x^{(1)}(s_{n-1}) - x^{(o)}(s_{n-1})\|_2^2 \, ds\, ds_1 \ldots ds_{n-1}$$

We note that both $s \to x^{(1)}(s)$ and $s \to x^{(0)}(s) = x_0$ *are* L^2 - continuous and hence bounded on $[t_0, T]$. Thus by using Cauchy integral formula, we get

$$\| x^{(n+1)}(t) - x^{(n)}(t)\|_2^2 \leq D_1^n \frac{(t - t_0)^n}{n!}, \tag{3.9}$$

where D_1 is a positive constant depending on t_0 and T. From the inequality (3.9), for any $n > r$, it follows that

$$\| x^{(n+1)}(t) - x^{(r+1)}(t)\|_2 = \| \sum_{m=r+1}^{n} (x^{(m+1)}(t) - x^{(m)}(t)\|_2$$

$$\leq \sum_{m=r+1}^{n} \left[D_1^m \frac{(t - t_0)^m}{m!} \right]^{1/2}$$

Thus, we observe that there is an $x(t)$ in $L^2(\ell)$.such that $x^{(n)}(t)$ converges uniformly to $x(t)$ in $L^2(\ell)$ on $[t_0, T]$. Since each $x^{(n)}(t)$ is L^2 - continuous and adapted, the process $x(t)$ is L^2 - continuous and adapted on $[t_0, T]$.

Finally we show that the process $x(t)$, $t \geq t_0$ satisfies equation (1.1).. It is clear that $x(t_0) = x_0$. Also

$$\| \int_{t_0}^{t} [F(s, x^{(n)}(s), A_1 x^{(n)}(s), A_2 x^{(n)}(s)) \tag{3.10}$$

$$- F(s, x(s), A_1 x(s), A_2 x(s))] \, ds \|_2^2$$

$$\leq 3K^2 (T - t_0) Q_1{}^* \int_{t_0}^{t} \| x^{(n)}(s) - x(s) \|_2^2 ds$$

$$\rightarrow 0 \text{ as } n \rightarrow \infty$$

since, for $s \, \varepsilon \, [t_0, T]$, $x^{(n)}(s) \rightarrow x(s)$ uniformly in $L^2(\ell)$. Further we note that

$$(3.11) \qquad \| \int_{t_0}^{t} [G(s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s))$$

$$- G(s, x(s), B_1 x(s), B_2 x(s))] \, dW(s) \|_2^2$$

$$\leq 3C_1{}^2 M^2 Q_2{}^* \int_{t_0}^{t} \| x^{(n)}(s) - x(s) \|_2^2 ds$$

$$\rightarrow 0 \text{ as } n \rightarrow \infty.$$

and

$$(3.12) \qquad \| \int_{t_0}^{t} dW(s) [H(s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s))$$

$$- H(s, x(s), P_1 x(s), P_2 x(s))] \|_2^2$$

$$\leq 3C_2{}^2 N^2 Q_3{}^* \int_{t_0}^{t} \| x^{(n)}(s) - x(s) \|_2^2 ds$$

$$\rightarrow 0 \text{ as } n \rightarrow \infty .$$

Thus from (3.2) and (3.10) - (3.12) it follows that

$$(3.13) \qquad x(t) = \lim_{n \rightarrow \infty} x^{(n+1)}(t)$$

$$= \lim_{n \rightarrow \infty} [x_0 + \int_{t_0}^{t} F(s, x^{(n)}(s), A_1 x^{(n)}(s), A_2 x^{(n)}(s)) ds$$

$$+ \int_{t_0}^{t} G(s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s))\, dW(s)$$

$$+ \int_{t_0}^{t} dW(s)\, H(s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s))\,]$$

$$= x_0 + \int_{t_0}^{t} F(s, x(s), A_1 x(s), A_2 x(s)) ds$$

$$+ \int_{t_0}^{t} G(s, x(s), B_1 x(s), B_2 x(s))\, dW(s)$$

$$+ \int_{t_0}^{t} dW(s)\, H(s, x(s), P_1 x(s), P_2 x(s))$$

This shows that $x(t)$, $t \geq t_0$ is a solution of the equation (1.1) with initial condition $x(t_0) = x_0$ for $t \in [t_0, T]$. Clearly $T > t_0$ is arbitrary, hence we can determine a solution $x(t)$ for all $t \geq t_0$.

We next prove the uniqueness of solutions of the equation (1.1). Let $x(t)$ and $y(t)$ be two L^2-continuous and adapted solutions of (1.1) such that $x(t_0) = y(t_0) = x_0$. By using $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, isometries (2.2) and (2.3), Schwartz inequality, hypothesis (H_1) and boundedness of $V_1, \dots V_n$ on $[t_0, t]$ we observe that

$$(3.14) \quad \|x(t) - y(t)\|_2^2 \leq 3(t - t_0) \int_{t_0}^{t} \| F(s, x(s), A_1 x(s), A_2 x(s))$$

$$- F(s, y(s), A_1 y(s), A_2 y(s)) \|_2^2\, ds$$

$$+ 3 \int_{t_0}^{t} \| G(s, x(s), B_1 x(s), B_2 x(s))$$

$$- G(s, y(s), B_1 y(s), B_2 y(s))\|_2^2 \, d\mu(s)$$

$$+ 3\int_{t_0}^{t} \| H(s, x(s), P_1 x(s), P_2 x(s))$$

$$- H(s, y(s), P_1 y(s), P_2 y(s))\|_2^2 \, d\mu(s)$$

$$\leq D \int_{t_0}^{t} \| x(s) - y(s)\|_2^2 \, ds,$$

Where D is as defined in (3.8). Thus from Gronwall's inequality it follows that

$$\|x(t) - y(t)\|_2^2 = 0, \quad t \varepsilon [t_0, T]. \tag{3.15}$$

Since both $x(t)$ and $y(t)$ are L^2- continuous, from (3.15) we get

$$\|x(t) - y(t)\|_2 = 0, \tag{3.16}$$

for all $t \in [t_0, T]$. Thus $x(t) = y(t)$, for each $t \in [t_0, T]$. This completes the proof of Theorem 1. The proof of Theorem 2 follows by arguments as in the proof of Theorem 1 with suitable modifications. We omit the details.

Proofs of Theorems 3 and 4

We define $y^{(n+1)}(t)$ and $z^{(n+1)}(t)$ for $n = 0, 1 \ldots$ with $y^{(0)}(t) = x_0$ and $z^{(0)}(t) = x_0^*$ similar to $x^{(n+1)}(t)$ with $x^{(0)}(t) = x_0$ defined in the proof of Theorem 1 for all $t \varepsilon I$. By using $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$, the isometries (2.2) and (2.3), the Schwarz inequality, the boundedness of the wave functions $v_1, \ldots, v_n$ over $[t_0, T]$, the hypothesis (H_1) we observe that

$$\| y^{(n+1)}(t) - z^{(n+1)}(t)\|_2^2 \leq 4 \|x_0 - x_0^*\|_2^2 \tag{4.1}$$

$$+ 4D^* \int_{t_0}^{t} \| y^{(n)}(s) - z^{(n)}(s)\|_2^2 \, ds,$$

where $D^* = D/3$ and D is as defined in (3.8). From (4.1) by successive iterations, we get

$$(4.2) \quad \| y^{(n+1)}(t) - z^{(n+1)}(t)\|_2^2 \leq 4 \sum_{m=0}^{n} \frac{(4D^*)^m}{m!} (t - t_0)^m \|x_0 - x_0^*\|_2^2$$

$$+ (4D^*)^{n+1} \int_{t_0}^{t} \int_{t_0}^{s_1} \dots \int_{t_0}^{s_n} \|y^{(0)}(s_{n+1}) - x^{(0)}(s_{n+1})\|_2^2 ds_{n+1} \dots ds_1$$

$$\leq 4 \sum_{m=0}^{n+1} \frac{(4D^*)^m}{m!} (t - t_0)^m \|x_0 - x_0^*\|_2^2 \leq 4 \|x_\upsilon - x_0^*\|_2^2 e^{4D^*T},$$

for all t, $t_0 \leq t \leq T$ and the result follows. this completes the proof of Theorem 3. The proof of Theorem 4 follows by similar argument as in the proof of Theorem 3 with suitable modifications, and hence we omit the details.

Remark 2.

We note that, one can very easily establish convergence theorems for the solutions of equations (1.1) and (2.5) similar to the Theorem 5.2 given in [2] with suitable modifications. In concluding this paper we note that regarding the study of stochastic differential equations involving Itô-Clifford integrals many problems are still open. The authors have tried to obtain a formula similar to Itô formula for stochastic differential equations involving Itô-Clifford integrals but could not get satisfactory answer. Once such a formula is available it will give rise to the study of many problems relating to stochastic equations involving Itô-Clifford integrals. The study of boundary value problems and various properties of second and higher order stochastic differential equations involving Itô-Clifford integrals awaits for its development.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp. 47-50

REGULARLY PERIODIC ELEMENTS OF A GROUP RING

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(Received 13.9.91)

ABSTRACT

In this note the author continues the study of regularly periodic elements of a group ring. A necessary and sufficient condition for an element to be regularly periodic in a group ring is obtained. For more properties about regularly periodic elements of a ring please refer [1].

Definition 1

Let R be an arbitrary associative ring. An element x of R is said to be a Regularly periodic element of periodicity n, $(n \geq 2)$ if there exists n distinct elements in R different from x say $r_1, r_2, \ldots, r_n$ $(r_1, r_2, \ldots, r_n \in R)$ such that

$$
\begin{aligned}
x^2 &= r_1 = x^{n+2} = x^{2n+2} = \ldots \\
x^3 &= r_2 = x^{n+3} = x^{2n+3} = \ldots \\
x^4 &= r_3 = x^{n+4} = x^{2n+4} = \ldots \\
&\cdot \\
&\cdot \\
&\cdot \\
x^n &= r_{n-1} = x^{2n} = x^{3n} = \ldots \\
x^{n+1} &= r_n = x^{2n+1} = x^{3n+1} = \ldots
\end{aligned}
$$

Throughout this paper K denotes a field and G a group: KG the group ring of G over K.

Proposition 1:

Let G be a finite group and K any field of characteristic zero. Then the group ring KG has regularly periodic elements.

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Proof.

Given G is finite, let $|G| = n$. *Consider* $\alpha = 1/n \; \Sigma g_i \, (g_i \varepsilon G)$ clearly $\alpha \in KG$ with $\alpha^2 = \alpha$. Hence KG has regularly periodic elements.

Proposition 2 :

Let G be any group having an element of finite order and K a field of characteristic zero. Then KG has a regularly periodic element. **Proof.** Let $g \varepsilon G$ be such that $g^m = 1 \; (m < \infty)$. Consider $\alpha = \frac{1}{m} \sum_{i-0}^{m-1} g^{l}$ in KG. Clearly α is a periodic element of KG.

Proposition 3.

Let G be a torsion free abelian group and K any field; No nontrivial element of KG is regularly periodic.

Proof.

G is torsion free abelian hence KG is a domain. Therefore KG cannot contain even an idempotent. Thus KG has no nontrivial regularly periodic element.

Proposition 4.

Let G be a finite group and K any field. If $H_1, H_2, \ldots, H_n$ are subgroups of G, then the group ring KG has periodic elements.

Proof.

For each subgroup H_i of G. $\alpha = \frac{1}{|H_i|} \Sigma h_i \left(h_i \varepsilon H_i\right)$ is a regularly periodic element of KG. (H_i) denotes the number of elements on H_i).

Proposition 5.

Let $Z_2 = (0, 1)$ and S_n be the symmetric group of degree n. Then every element of Z_2S_n which is not nilpotent is regularly periodic.

Proof.

Since order of Z_2S_n is $2^{n!}$ Z_2S_n is a finite ring and hence every non nilpotent element is regularly periodic.

Theorem 6.

Let G be an abelian group. The group ring KG has a nontrivial regularly periodic element if and only if G has an element of finite order.

Proof.

If G has an element of finite order by Proposition 2 the group ring KG has a regularly periodic element.

Conversely if KG has a regularly periodic element to prove G has an element of finite order. Let $\alpha = \Sigma\alpha_i g_i \,\varepsilon\, KG$ be regularly periodic; this implies there exists n distinct elements $\beta_1, \beta_2, \ldots\ldots\ldots\ldots \beta_n \,\varepsilon\, KG$ with

$$\alpha^2 = \beta_1 = \alpha^{n+2} = \alpha^{2n+2} = \ldots$$
$$\alpha^3 = \beta_2 = \alpha^{n+3} = \alpha^{2n+3} = \ldots$$
$$\alpha^4 = \beta_3 = \alpha^{n+4} = \alpha^{2n+4} = \ldots$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$\alpha^n = \beta_{n-1} = \alpha^{2n} = \alpha^{3n} = \ldots$$
$$\alpha^{n+1} = \beta_n = \alpha^{2n+1} = \alpha^{3n+1} = \ldots$$

Clearly $\alpha^2 (\alpha^n - 1) = 0$ this implies $\alpha^n = 1$ or KG has divisors of zero. Now KG has no divisor of zero only when G is not torsion free. That is there are elements in G of finite order. $\alpha^n = 1$ is impossible if every element in G is torsion free thus G has elements of finite order. Hence the theorem.

Remark .

If we do not assume G is abelian we may not be able to conclude KG has no zero divisors if every element of G is torsion free. Thus the zero divisor conjecture for group rings can be reformulated as follows.

Conjecture.

Let G be a torsion free non abelian group and K any field. Can KG have regularly periodic elements?

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 51-54

FIXED POINTS FOR A PAIR OF DENSIFYING MAPPINGS

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(Received 24 - 9- 1991)

ABSTRACT

In the present paper we establish a fixed point theorem for a pair of densifying mappings which includes the results or Khan and Fisher [2] and Furi and Vignoli [1].

Keywords and Phrases : Fixed point, densifying group,

Mathematics Subject classifications (1991) : 54 H25, 47 H10

INTRODUCTION

Kuratowski [3] introduced the concept of **measure of non-compactness** of a bounded set in a metric space as follows :

Defination 1 : Let A be a bounded set in a metric space (X,d). Then the **measure of non-compactness** of A, written as $\alpha(A)$, is the infimum of all $\varepsilon > 0$ such that A admits a finite covering by sets with diameter less than ε.

Definition 2 : [1] A mapping T of a metric space (X,d) into itself is said to be **densifying** if for every bounded subsets A of X with $\alpha(A) > O$.

$$\alpha(T(A)) < \alpha(A)$$

Reich [4] obtained the following theorem.

Theorem A : Let T be a mapping of a compact metric space X into itself such that

$$d(Tx, Ty) < (1/2)[d(x, Tx) + d(y, Ty)]$$

for all $x,y \in X$ with $Tx \neq Ty$, then T has a unique fixed point.

MAIN RESULT

We present the following theorem.

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Theorem : Let S and T be continuous densifying mappings of a **complete metric space** (X,d) into itself such that for all x,y in X

$$F_1(Sx,TSy) < max\{[F_2(x,Sy),F_2(x,Sx),F_2(y,Sy),(1/2)F_2(x,Sx)$$

$$+ (1/2)F_1(Sy,TSy)]\ U\ min\ [F_2(x,TSy),\ F_2(Sy,Sx)]\}$$

and

$$F_2\,(Tx,STy) < max\{[F_1(x,Ty),F_1(x,Tx),F_1(y,Ty),(1/2)F_1(x,Tx)$$

$$+ (1/2)\ F_2(Ty,STy)]\ U\ min[F_1(x,STy),F_1(Ty,Tx)]\}$$

where F_1 and F_2 are real-valued functions of $X \times X$ into $[0, \infty)$ with either F_1 or F_2 lower semi-continuous. If for some x_o in X, the sequence $\{x_n\}$ defined by $Sx_{2n} = x_{2n+1}$, $Tx_{2x+1} = x_{2n+2}$ for $n = 0,1,2,\ldots\ldots.$ is bounded and $F_1\,(x,x) = F_2(x,x) = 0$ for all $x \in X$, then either S or T has a fixed point z. Further, if z is a common fixed point of S and T, then it is the unique common fixed point of S and T.

Proof : We may assume that F_1 is lower semi-continuous. Consider the set $A = \{x_{2n+1} : n = 0,1,2,\ldots\}$. Then $ST(A) = \{x_{2n+1} : n = 1,2,\ldots\ldots.\}$ and so $ST(A) \subset A$. The continuity of S and T now implies that $ST(\overline{A}) \subset \overline{ST(A)}$. Now suppose that $\alpha\ (A) > 0$. Then

$$\alpha\ (\mathrm{A}) = \alpha\ (\{x_1\}\ U\ ST(A))$$

$$= max\ \{\ \alpha\ (x_1),\ \alpha\ (ST(A))\}$$

$$= \alpha\ (ST(A)) < \alpha\ (A)$$

giving a contradiction. This contradiction implies that $\alpha\ (A) = 0$ and so A is relatively compact. Since, the function F_1 is lower semi continuous, the function f defined by

$$f(x) = F_1\ (x,Tx)$$

for all x in $\overline{A}$ is lower semi-continuous and so assumes its minimum value at some z in $\overline{A}$. Thus if $Tz \neq z$ and $STz \neq Tz$ then $STz \in \overline{A}$ and

$$f(STz) = F_1\ (STz,\ TSTz)$$

$$< \max \{[F_2 (Tz, STz), F_2 (Tz, STz), F_2 (Tz, STz), (1/2) F_2 (Tz, STz)$$

$$+ (1/2) F_2 (STz, TSTz)] \text{ U } \min [F_2(Tz, TSTz), F_2(STz, STz)]\}$$

$$= \max \{F_2(Tz, STz), (1/2) F_2 (Tz, STz) + (1/2) F_1 (STz, TSTz)\}$$

We have either

$f(STz) < F_2(Tz, STz)$ or $f(STz) < (1/2) F_2 (Tz, STz) + (1/2) F_1 (STz, TSTz)$

which implies

$f(STz) < F_2 (Tz, STz)$

Therefore

$$f(STz) < F_2(Tz, STz)$$

$$< \max \{[F_1(z,Tz), F_1 (z,Tz), F_1 (z,Tz), (1/2)F_1(z, Tz)$$

$$+(1/2)F_2 (Tz, STz)] \text{ U } \min [F_1(z,STz), F_1 (Tz,Tz)]\}$$

$$= \max\{F_1(z,Tz), (1/2)F_1 (z,Tz) +(1/2) F_2 (Tz, STz)\}$$

which implies

$$f(STz) < F_1 (z,Tz) = f(z)$$

contradicting the definition of z. It follows that either $Tz = z$, in which case z is a fixed point of T, or $STz = Tz$, in which case $z' = Tz$ is a fixed point of S.

Now suppose that z is a common fixed fixed point of S and T. Then if S and T have a second distinct common fixed point z' then $z' \neq Sz'$, $z \neq Tz'$, and so.

$$F_1 (z,z') = F_1(Sz,TSz')$$

$$< \max \{[F_2(z,Sz'), F_2(z,Sz), F_2(z',Sz'), (1/2) F_2(z,Sz)$$

$$+(1/2) F_1 (Sz', TSz')] \text{ U } \min [F_2(z,TSz'), F_2 (Sz',Sz)]\}$$

$$= F_2(z,Sz')$$

$$= F_2 (Tz, STz')$$

$$< \max \{[F_1(z,Tz'), F_1(z,Tz), F_1 (z', Tz'), (1/2) F_1(z,Tz)$$

$$+ (1/2) F_2(Tz',STz')] \text{ U } \min [F_1 (z,STz'), F_1(Tz',Tz)]\}$$

$= F_1(z,z')$

giving a contradiction. The common fixed fixed point z must therefore be unique. This completes the proof of the theorem.

Corollary 1 [2] : Let S and T be continuous densifying mappings of a complete metric space (X,d) into itself such that for all x,y in X

$$F(Sx, TSy) < F(x,Sy),\ x \neq Sy$$

and

$$F(Tx, STy) < F(x,Ty),\ x \neq Ty,$$

where F is a real valued lower semi-continuous function fo XxX into $[o, \infty]$. If for some x_0 in X, the sequence $\{x_n\}$ defined by $Sx_{2n} = x_{2n+1}$, $Tx_{2n+1} = x_{2n+2}$ for $n = 0,1,2,....$ is bounded, then either S or T has a fixed point z. Further, if z is a common fixed point of S and T, then it is the unique fixed point of S and T.

Corollary 2 [1] : Let T be a continuous densifying mapping of a complete metric space (X,d) into itself and let F, be a real valued lower semi-continuous function of XxX into $[0, \infty]$ such that for all x,y in X.

$$F(Tx,Ty) < F(x,y),\ x \neq y$$

If for some x_0 in X the sequence $\{x_n\}$ defined by $Tx_{2n} = x_{2n+1}$ for $n = 0,1,2,......$ is bounded, then T has a unique fixed point z.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 55-58

AN ALTERNATIVE ESTIMATOR FOR THE MEAN OF FINITE POPULATION USING AUXILIARY VARIABLES

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(Received: 30-03-1991, after revision: 23-10-1991)

ABSTRACT

The supplementary information on auxiliary variables is very useful at the estimation stage in sample surveys. Several estimators developed using the supplementary information on the auxiliary variables are available in the literature. In this paper an alternative estimator of the population mean ($\bar{Y}$) using supplementary information on two auxiliary variables x (positively correlated with y) and z (negatively correlated with y) has been suggested. In section 2 the bias and the mean squared - error (MSE) of the proposed estimator have been obtained and in section 3 efficiency of the proposed estimator has been compared with the usual unbiased estimator $\bar{y}$ and with that of Srivastava [3] under simple random sampling with replacement (SRSWR).

Mathematics Subject Classification (1991) : 62 D 05

Keywords: Auxiliary variable, mean square error, ratio estimator, simple random sampling with replacement (SRSWR), unbiased estimator

INTRODUCTION

In large sample surveys, data on more than one auxiliary variables are often collected. Some of these variables may be correlated with the main character under study (y). Several successful attempts have been made to use multi-auxiliary information in increasing the precision of estimator of the population mean $\bar{Y}$. Olkin [1] extended the ratio method of estimation to incorporate supplementary information on several auxiliary variates each positively correlated with y. The case of multi-auxiliary variates, some positively correlated with y and the others negatively correlated with it, has been discussed by Srivastava [3,4] and Rao and Mudholkar [2].

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Let a simple random sample of size n is drawn from a population of size N. Let $\overline{Y}$, $\overline{X}$ and $\overline{Z}$ be the population and $\bar{y}$, $\bar{x}$ and $\bar{z}$ be the sample means of the characters y, x (positively correlated with y) and z (negatively correlated with y) respectively. Two estimators of the population mean $\overline{Y}$ based on the auxiliary information are well known in the literature, namely, the ratio estimator $\bar{y}(\overline{X}/\bar{x})$ and the product estimator $\bar{y}\ (\bar{z}/\overline{Z})$. Srivastava [3], however, suggested an estimator for $\overline{Y}$ using the supplementary information on both the variables x and z, as

$$\bar{y}_{Rw} = \bar{y}\ [w_1 (\overline{X}/\bar{x}) + w_2 (\bar{z}/\overline{Z})] \tag{1}$$

where w_1 and w_2 are the weights associated with the two estimators such that $w_1 + w_2 = 1$ and studied its properties.

Further let $\bar{x}^*$ and $\bar{z}^*$ be the means corresponding to the $(N - n)$ units no included in the sample. An alternative estimator ($\bar{y}_{Rw'}$) of $\overline{Y}$ associated with x and z can be constructed which is linear in $(u\ /U)^{\delta u}$ where u represents x, z; U represents X, Y and $\delta_u = sign\ (\ \rho_{yu})$. In other words.

$$\bar{y}_{Rw'} = \bar{y}[\ w'_1 (\bar{x}^* /\ \overline{X}) + w'_2 (\overline{Z}\ /\ \bar{z}^*\ \ )] \tag{2}$$

Where w_1 and w_2 are the weights such that $w'_1 + w'_2 = 1$

BIAS AND MEAN SQUARED ERROR OF $\bar{y}_{Rw'}$

Following Sukhatme and Sukhatme [5], the bias and the MSE of the estimator $\bar{y}_{Rw'}$ to the order of $O\ (n^{-1})$ can easily be seen to be,

$$B(\bar{y}_{Rw'}) = g\overline{Y}\,n^{-1}[w'_2 (g\ C_z^{\,2} + C_{zy}\ ) - w'_1 C_{xy}\ ] \tag{3}$$

and

$$\begin{aligned} MSE\ (\bar{y}_{Rw'}) = \ & \overline{Y}^{\,2} n^{-1}[w_1'^{\,2} (g^2 C_x^{\,2} + C_y^{\,2} - 4gC_{xy}) + w'^{\,2}_2 (3g^2 C_z^{\,2} + C_y^{\,2} + 4gC_{zy}) \\ & + 2w_1' w_2' (g^2\ C_x^{\,2}\ -\ g^2\ C_{xz}^{\,2} +\ C_y^{\,2} + 2gC_{zy} -\ 2gC_{xy}) \\ & -2w'_2 (g^2 C^2_z + gC_{zy}) + 2w_1'\ g\ C_{xy}\ ] \end{aligned} \tag{4}$$

where,

$$g = \ n\ /(N\text{-}n),\ C_x^{\,2} = \sigma_x^2\ /\ \overline{X}^{\,2},\ C_y^{\,2} = \sigma_y^2\ /\ \overline{Y}^{\,2},\ C_z^{\,2} = \sigma_z^2\ /\ \overline{Z}^{\,2},$$
$$C_{xy} = \rho_{xy}\ C_x C_y,\ C_{zy} = \rho_{xy} C_z C_y,\ \ C_{xz} = \rho_{xz} C_x C_z\,,$$

$$\rho_{xy} = Cov(x,y)/\sigma_x\sigma_y, \quad \rho_{yz} = Cov(y,z)/\sigma_y\sigma_z, \quad \rho_{xz} = Cov(x,z)/\sigma_x\sigma_z$$

The optimum weights obtained by minimizing (4) subject to the condition that $w'_1 + w'_2 = 1$ can be seen to be

$$w'_1 = \frac{g(C_z^2 + C_{xz}) + C_{xy} + C_{yz}}{g(C_x^2 + C_z^2 + 2C_{xz})} \quad \text{and}$$

$$w'_2 = \frac{g(C_x^2 + C_{xz}) + C_{xy} + C_{yz}}{g(C_x^2 + C_z^2 + 2C_{xz})} \tag{5}$$

The optimum values of bias and MSE can be obtained by substituting the values of w'_1 and w'_2 in equations (3) and (4) as,

$$B_0(\bar{y}_{Rw'}) = \frac{g\overline{Y}}{n}\left[\left(gC_z^2 + C_{zy}\right) - \frac{\{g(C_z^2 + C_{xz}) + C_{xy} + C_{yz}\}(gC_z^2 + C_{zy} + C_{xy})}{g(C_x^2 + C_z^2 + 2C_{zx})}\right] \tag{6}$$

and

$$M_0(\bar{y}_{Rw'}) = \frac{\overline{Y}^2}{n}\left[\left(g^2C_z^2 + C_y^2 + 2gC_{zy}\right) - \frac{\{g(C_z^2 + C_{xz}) + C_{xy} + C_{yz}\}^2}{g(C_x^2 + C_z^2 + 2C_{xz})}\right] \tag{7}$$

In practice, the population coefficients of variation (C_y, C_x, C_z) and the correlation coefficients (ρ_{xy}, ρ_{zy} and ρ_{xz}) are not known so that optimum weights will have to be estimated from the sample values. However, what may be known in practice may be the nature of the correlation between the variables y, x and z. This knowledge is essential for constructing the estimator. The sample estimates of optimum weights can be obtained by replacing each term in (5) by their sample estimates.

For the sake of simplicity, we may assume that

$$C_x = C_z \text{ and } \rho_{yx} = -\rho_{yz} \tag{8}$$

so that

$$w'_1 = w'_2 = 1/2 \tag{9}$$

From (6), (7), (8) and (9) the optimum bias and MSE of $\bar{y}_{Rw'}$ to the order of $O(n^{-1})$ can be found to be,

$$B^*_0(\bar{y}_{Rw'}) = g\overline{Y}(2n)^{-1} C_x(gC_x - 2\rho_{xy}C_y) \tag{10}$$

and

$$M_0^*(\bar{y}_{Rw'}) = \overline{Y}^2 n^{-1}[C_y^2 + (g^2C_x^2/2)(1-\rho_{xz}) - 2gC_xC_y\rho_{xy}] \tag{11}$$

The estimator $\bar{Y}_{Rw'}$ is unbiased when $g = 2\rho_{yx}C_y/C_x$

It may be noted that when $w_1 = w_2 = 1/2$, Srivastava [3] obtained the minimum MSE of the estimator $\bar{y}_{Rw'}$ as,

$$M_0(\bar{y}_{Rw}) = \bar{Y}^2/n^{-1}\,[C_y^2 + (C_x^2/2)(1-\rho_{xz}) - 2\rho_{xy}\,C_xC_y] \qquad (12)$$

EFFICIENCY COMPARISONS

The variance of the usual unbiased estimator $\bar{y}$ is given by

$$V(\bar{y}) = \bar{Y}^2C_y^2/n$$

From (11), (12), and (13) it can easily be shown that $\bar{y}_{Rw'}$ will be more efficient than $\bar{y}$ and $\bar{y}_{Rw}$ when

$$g/4 \le \rho_{yx}\,C_y/\{(1-\rho_{xz})\,C_x\} \le (1+g)/4$$

Which is true when $(1-g) \ge 0$, that is when $n \le N/2$. This condition is quite obvious as the proposed estimator is dual of the estimator proposed by Srivastava [3].

ACKNOWLEDGEMENT

The author is thankful to the referee for his constructive comments on the earlier version of the paper

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 59-66

RELIGIOSITY AND LOCUS OF CONTROL AMONG COLLEGE STUDENTS

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(Received 25-11-1991)

ABSTRACT

Present investigation attemptes to study religiosity and locus of control among college students. The sample comprises of 200 college students (100 males and 100 females)po selected from Hardwar District by incidental sampling. Results reveal that female are more religious than male students, and sex differences exist in regard to locus of control among college students.

Mathematical Subject classifications (1991) : 92 J 99

Keywords & phrases : Religiosity,

INTRODUCTION

Religion is a powerful institution which plays an important role in shaping the social behaviour. In the nineteenth and twentieth centuries numerous ideas have been put forward regarding psychological and sociological factors that are responsible for the origin of religion.

The most important of these are (i) the Marxian theory which maintains that religion is one of the ideological reflection of the current state of economic interactions in a society, (ii) the similar, but more elaborately developed theory of the sociologist Emile Durkheim holds that religious beliefs constitute a projection of the structure of society, and (iii) the Freudian theory which emphasises that religious beliefs arise from projections designed to alleviate certain kinds of unconscious conflicts (7).

Galloway (8) has defined religion as the faith in a power beyond himself whereby he seeks to satisfy emotional needs and gain stability of life, which he expresses in the act of worship and service. The real religion is equally acceptable to all minds, and is almost equally philosophic, emotional, mistic and conducive to action. Williams and Cole (31) found in their study that

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religious persons carry less feeling of insecurity than non-religious persons. Mayo, et al. [16] concluded on the basis of their study that the religious men are significantly less depressed, less schizophrenic and less psychopathic deviant than non-religious ones. Adjustment and social responsibility were found to be possible correlates of religiousness (cf. [32]). Tiwari, Singh and Srivastava [28] found that high religious value oriented persons have better pattern of adjustment than low religious value oriented persons. In a study, Bhushan [3] found that religious girl students were significantly better adjusted than non-religious girl students.

Hassan and Khalique [10] showed that muslims tended to have high degree of religiousity in comparision to Hindus. Status (high/low), and sex (male/female) did not influence the religious attitudes of either Hindus or Muslims. Religiosity was found to have significant positive correlations with anxiety, rigidity and intolerance of ambiguity. Verma and Upadhyay [30] found that high religious students showed less social distance with others, if ethnic groups were treated as a whole. However, the students in general did show quite different attitudes towards various ethnic groups, and this is equally true with low and high religiosity groups, when treated separately. Sinha and Jha [27] concluded that correlation between religiosity and the total as well as the area-wise scores of religious prejudices revealed no significant relationship between the two, and high and low religous persons did not differ significantly on their religious prejudice scores.

The concept of locus of control was developed by Phares [19] in relation to beliefs regarding internal versus external control of reinforcement. It is assumed that individuals develop a general expectancy regarding their ability to control their lives. People who believe that the events that occur in their lives are a result of their own behaviour and/or personality characteristics are said to have an "expectancy of internal control", while people who believe events in their lives to be a function of luck, chance, fate, powerful others or powerful beyond their control or comprehension are said to have an "expectancy of external control".

Rotter [22] provides a useful tool for measuring individual differences to the extent to which reinforcement is viewed as a consequence of one's own behaviour or a consequence of such forces as chance, fate or powerful others. Singh [24] found that socio-culturally deprived students were significantly more external and chance oriented as compared to non-deprived students. Significant differences between internal - external locus of control in group conformity behaviour were found by Singh [25]. The results of the study conducted by Rao, et al. [20] suggested that the relationship between locus of control orientation and adjustment was not very clear. Locus of control orientation was found to determine the use of specific coping behaviours, but did not play a significant role in determining the experience and perception of stressful life events. Several studies have been carried out on religiosity, locus of control and other related variables (see, for instance, [1], [4], [9], [12], [14], [15], [23], [29] and [33]. The present paper reports data pertaining to the religiosity and locus of control among male and female college students.

Hypotheses :

(i) There would be significant difference between male and female students in terms of religiosity.
(ii) There would be significant difference between male and female students in terms of locus of control.

Sample :

The sample comprises of 100 male and 100 female students studying in class XI and XII in Hardwar District. The sample was selected by incidental sampling technique.

Scales :

Bhushan's Religiosity scale [2] and Rotter's Locus of control scale as standardized by Kumar and Srivastava [13] were used to measure the two variables of religiosity and locus of control among college students.

RESULTS AND DISCUSSION

The results confirm the contention of the hypothesis of significant difference between male and female students on religiosity scores (Table 1). The female students with a mean of 128.39 are found to be more religious in comparison to male students with a mean score of 121.62. This difference may be attributed possibly to cultural training and child rearing practices in the society. These appears to be a different role expectations in regard to boys and girls. Besides the temperamental difference also attracts girls towards religion. Similar conclusions may be found in the investigations conducted by Helode and Barlinge [11], and Bhushan and Singh [4].

Table - 1

Showing Mean and Standard Deviatioin of Male and Female Students in Terms of Religiosity

Students	N	Religiosity Scores		
		Mean	S.D.	t-value
Male	100	121.62	17.34	2.82
Female	100	128.39	16.58	P < 0.01

Table - 2

Showing Mean and Standard Deviatioin of Male and Female Students in Terms of Locus of Control.

Students	N	Locus of control scores		
		Mean	S.D.	t-value
Male	100	9.96	3.13	3.40
Female	100	11.56	3.43	P < 0.01

The results in regard to locus of control confirm that the female students are more external in their locus of control orientation than boys (Table 2). Female students are found to have developed a high "expectancy of external control". They believe that the events in their lives are controlled by such factors as luck, chance and super natural powers which are beyond there control. On the other hand the male students have greater faith in their own abilities and believe that their own abilities and efforts determine and shape their future.

The genesis of this difference can be directly traced in the amount of faith which boys and girls have in religion as discussed earlier, religion is a "faith in a power beyond himself; Since girls are more religious they have greater faith in super natural powers.

These findings are in line with other studies reported by Person and Schneider [18], Blass [5] and Chandran [6] and Kunhikrishnan and Mathew [12] on sex differences in Internal-External locus of control. Results of Helode and Barlinge [11] indicate that externality and religiosity are positively correlated. Females are more religious than males. Molinari and Khanna [17] have suggested that adopting an external orientation when situations are uncontrollable as a defense against negative self-evaluation is a positive characteristic. Such persons attribute failure to luck as a defensive manoeuvre, and yet, resemble internals on behavioural indices of task performance.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 67-74

ON THE SOLUTIONS OF VISCOUS INCOMPRESSIBLE FLUID FLOW

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(Received 13-12-1991)

ABSTRACT

A technique has been developed to combine two or more known solutions of uniform flow of viscous - incompressible fluid in pipes to a single new solution. Also in the light of the descussion a very bold proposition in the form of a principle has been put forward. This principle has been named as ***the principle of second degree harmonics***. The complete study has been divided into two sections. Section I deals with the formation of a new technique to combine two or more known solutions of viscous - incompressible fluid - flow to a single one and its applications. Section II deals with a particular case, how to derive a solution for a regular figure of n-sides from the solution for an equilateral triangle.

Mathematics Subject Classifications (1991): 76

Keywords and Phrases: Compoundable, viscous - incompressible fluid, Boussincsq's' formula.

SECTION I

In a problem of uniform flow of viscous - incompressible fluid in pipes, the boundary along which the velocity vanishes is a closed figure. Any section of this closed figure by a chord, we shall call a segment. Every chord divides the figure into two segments.

Let us consider two such segments taken from two different solutions and they are named as A and B, the bounding chord being PQ and RS (fig. 1).

We shall enquire into necessary and sufficient conditions under which A and B may be regarded as two segments of the same pipe so that by combining the two segments we may arrive at a new solution. The conditions must be :

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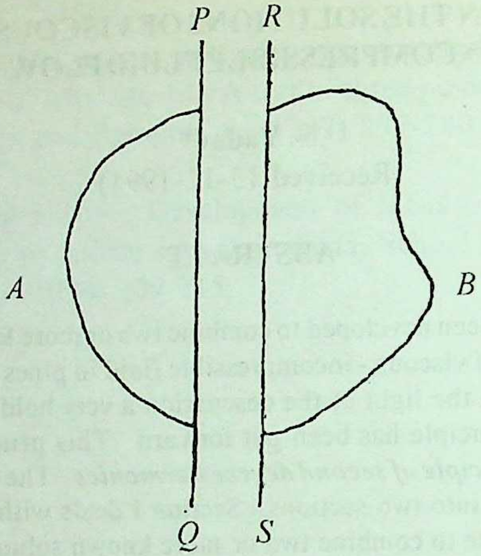

Fig. 1

Geometrical Ccondition :

Chord PQ must be equal to RS

Mechanical Ccondition :

The velocity function in A at every point of PQ must have the same value as the velocity function in B at the corresponding point in RS. By correspondence we mean the coincidence in superposition of PQ over RS.

The mechanical condition guarantees that along the planes of PQ and RS the flow will be the same at every point, so the flow on either side of this common plane will not be disturbed.

Two segments satisfying the geometrical and mechanical conditions, we shall call *compoundable*.

Segments of two known solutions may be joined to give a new solution if they are compoundable. A segment is always compoundable to its mirror image for if D is the mirror image of C as shown in Fig. (2), then obviously the geometrical and mechanical conditions are satisfied.

Now we proceed to consider instead of segments the sector bounded by

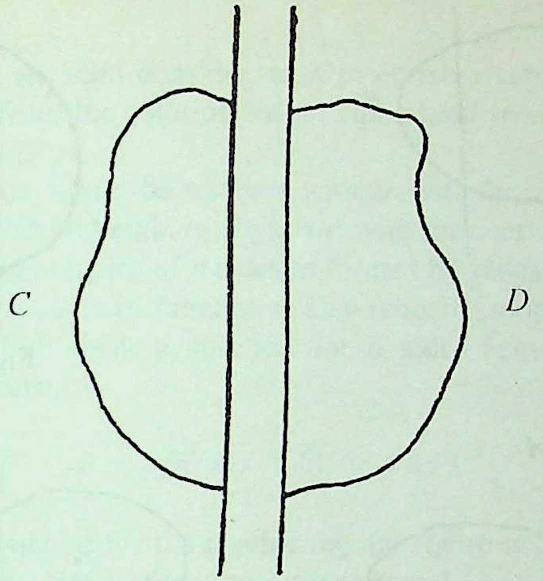

Fig. 2

radii vector making an angle $2\pi/n$, n being an integer. If n sectors be placed in such a way that each is the mirror image of adjoining one, the vertices coinciding, then the geometrical and mechanical conditions for compoundability are satisfied. Only if n is odd the two radii vector must be of equal length. When n is even they may be of unequal length.

We shall now consider some applications of the technique developed above. Considering the segments of an ellipse. Let A be the major segment containing the point of maximum velocity and B, a minor segment (Fig. 3). By combining each with its mirror image we arrive at two solutions L and M (Fig. 4 & 5) respectively. M is specially notable for its peculiarity of having two points of maximum velocity. By taking a major segment of M as indicated by dotted lines, we may get a solution with four points of maximum velocity as shown in Fig. 6. There is no limit to the variety of solutions that can be derived from one known solution. Next, let us consider a sector bounded by radii vectors making angle $2\pi/n$ n being an integer. Taking again the ellipse $PLQN$, let us form two sectors, one major sector containing the centre 0, and the other a minor sector, by two equal intersecting chords PQ and LN, intersecting at a point M on the major axis (Fig. 7).

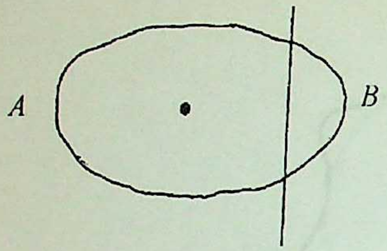

Fig. 3

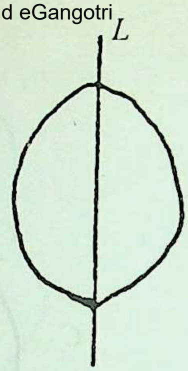

Fig. 4

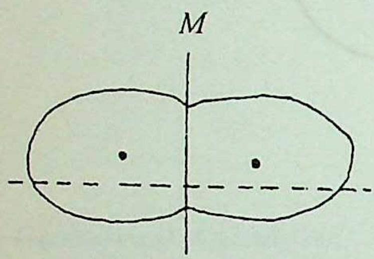

Fig. 5

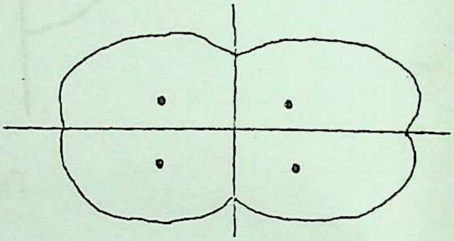

Fig. 6

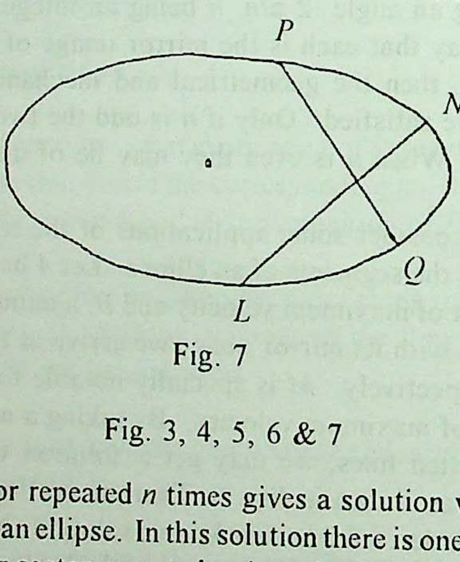

Fig. 7

Fig. 3, 4, 5, 6 & 7

The minor sector repeated *n* times gives a solution when the boundary consists of *n* arcs of an ellipse. In this solution there is one point of maximum velocity. The major sector repeated *n* times is more remarkable as it gives a solution with *n* points of maximum velocity.

In all such cases the area, the total flow per unit time, average velocity, maximum velocity, *k* and *k′*, are calculated by considering a single segment or sector and multiplying by the necessary integer.

SECTION II

In this section we shall consider, how to derive a solution for regular figure of *n* sides, from the solution for an equilateral triangle.

The best method would be to form a sector with the centre *O* as vertex and radii vectors *OP, OQ* making angle π/n with the central axis as shown in Fig. 1. The regular figure of *n* sides is formed by repeating the sector *n* times round *O*. The velocity function is also repeated in the same manner. In this way we shall form a solution for n sided regular figure from Boussinesq's formula,

$$w = - (K/4b)\ (y-b)\ (y^2 - 3x^2)$$

The velocity function for the *n* sided regular figure will be obtained by repeating the above velocity function *n* times round *O'* as shown in Fig. 2.

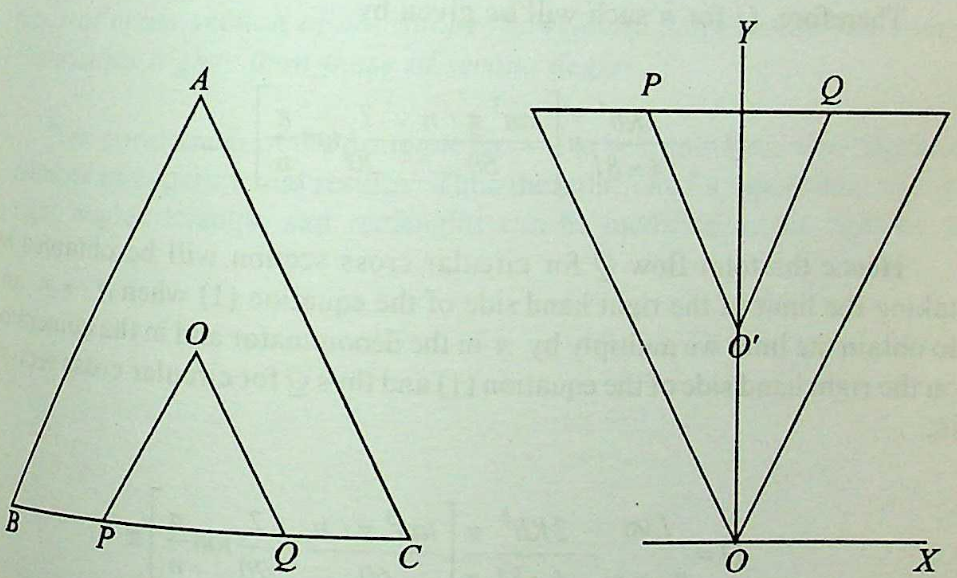

Fig. 1

Fig. 2

Fig. 1 & 2

We shall calculate the total flow in the sector $PO'Q$ and thus get the total flow for the regular n sided figure.

For the sector $PO'Q$, total flow Q is given by,

$$2\int_{\theta=0}^{\theta=\frac{\pi}{n}}\int_{r=0}^{r=\frac{b}{3\cos\theta\, k}} -\left(K\big/_{4b}\right)[r^3\cos^3\theta - 3r^3\cos\theta\sin^2\theta + br^2\cos^2\theta +$$

$$br^2\cos^2\theta + br^2\sin^2\theta - 4b^3\big/_{27}]\, rdrd\theta$$

$$2\int_0^{\pi/n} -\frac{Kb^4}{4\times 81}\left[\frac{1}{20\cos^4\theta} - \frac{2}{5\cos^2\theta}\right]d\theta$$

$$Q = -\frac{Kb^4}{4\times 81}\left[\frac{\tan^3\pi/n}{60} - \frac{7}{20}\tan\frac{\pi}{n}\right]$$

Therefore, Q for n such will be given by

$$Q = -\frac{2Kb^4}{4\times 81}n\left[\frac{\tan^3\pi/n}{60} - \frac{7}{20}\tan\frac{\pi}{n}\right]$$

Hence the total flow Q for circular cross section will be obtained by taking the limit of the right hand side of the equation (1) when $n\to\infty$ and to obtain the limit we multiply by π in the denominator and in the numerator on the right hand side of the equation (1) and thus Q for circular cross section is,

$$Q = \lim_{n\to\infty} -\frac{2Kb^4}{4\times 81}\frac{\pi}{n}\left[\frac{\tan^3\pi/n}{60} - \frac{7}{20}\tan\frac{\pi}{n}\right]\pi$$

$$Q = \frac{2Kb^4}{4\times 81}\cdot\frac{7}{20}\pi = \frac{kb^4}{\frac{40}{7}}\pi$$

Since $a = b/3$; a will be the radius of the circular cross section. We thus see that the total flow in tube of circular cross section derived from *Boussinesq's* formula for equilateral triangle does not come equal to the usual value, but is *greater* in the ratio 7:5.

It has been established by the author in his paper' *A generalized study of a viscous incompressible fluid flow through various cross sections of a tube*' that the formula for the total flow in circular tube, when derived from the formula for rectilinear figure, agrees with the usual value. One thing of remarkable importance is that both the usual velocity function for circular cross section and our formula for rectilinear figures, contain harmonics of second degree only. *Boussinesq's* formula, however, contains harmonics of third degree.

There does not seem to be any flaw in the discussion at section (I), so we are tempted to put forward a very bold proposition in the form of a principle. This principle we shall call the *principle of second degree harmonics*. The principle may be stated in the following words.

In all cases of Laminar flow of viscous incompressible fluids through tubes of cross section of any shape the velocity function can not contain Harmonics higher than those of second degree.

The correctness of the principle has yet to be established from sufficient number of experimental results. Thus the solutions for equilateral triangle, right angled triangle and rectangles can be modified in the light of this principle.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 75-82

VEDIC GEOMETRY†

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(Received 23-12-1991)

ABSTRACT

This talk intends to discuss *Śulbas'* period, and give some glimpses from *Brāhmaṇas* and *Śulbas'* concerning the origin of Pythagorian triplets and the irrationality concept of real numbers.

Mathematics Subject Classifications (1991) : O1A32

Keywords and Phrases: *Śulba Sūtra*, *trita*, pythagorian triplets

INTRODUCTION

According to the *Jaina* canon, geometry is the lotus of Mathematics (see *Sūtrakrta* (II. 1. 154), cf. (10), p. XII). Ancient Indian savants are now credited for fundamental pioneering work in geometry A very deep analysis by Seidenberg (See [21], p. 108) concludes "... Greek geometry (especially the Theorem of Pythagoras) did not somehow make its way into *Vedic* geometry, as Greek geometry is only supposed to have started about 600 B.C.".

First I quote the following *mantras* from *Yajurveda* (see [17], p. 210):

एको॑ च मे ति॒स्रश्च॑ मे ति॒स्रश्च॑ मे पञ्च॑ च मे॒ पञ्च॑ च म स॒प्त च मे स॒प्त च मे॒ नव॑ च मे॒ नव॑ च म॒ऽएका॑दश च म॒ऽएका॑दश च मे॒ त्रयो॑दश च मे॒ त्रयो॑दश च मे॒ पञ्चद॑श च मे॒ पञ्च॑दश च मे स॒प्तद॑श च मे स॒प्तद॑श च मे॒ नव॑दश च मे॒ नव॑दश च म॒ऽ एक॑विँ॒शतिश्च म॒ऽ में त्रयो॑विँ॒शतिश्च मे॒ त्रयो॑विँ॒शतिश्च मे॒ पञ्च॑विँ॒शतिश्च मे॒ पञ्च॑विँ॒शतिश्च मे स॒प्तविँ॒॑शतिश्च मे स॒प्तविँ॒॑शतिश्च मे॒ नव॑विँ॒शतिश्च मे॒ नव॑विँ॒शतिश्च म॒ऽएकत्रिँ॒शच्च म॒ऽएकत्रिँ॒शच्च मे॒ त्रय॑स्त्रिꣳशच्च मे य॒शेन॑ कल्पन्ताम् ॥१ ॥

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†Talk presented at National Seminar on Science in Ancient India, Kumaun Univ., Nainital, Oct. 12-16, 1991.

च तं स्रश्च मेऽष्टौ च मेऽष्टौ चं मे द्वादश च मे द्वादश च मे षोडश च मे
षोडश च मे विंशतिश्चं मे विंशतिश्चं मे चतुर्विंशतिश्च मे चतुर्विंशतिश्च
मेऽष्टाविंशतिश्च मे द्वात्रिंशच्च मे द्वात्रिंशच्च षट्त्रिंशच्च मे षट्त्रिंशच्च मे
चत्वारिंशच्च मे चत्वारिंशच्च मे.चतुश्चत्वारिंशच्च मे चतुश्चत्वारिंशच्च
मेऽष्टाचत्वारिंशच्च मे यशेनं कल्पन्ताम् ॥२ ॥
य० अ० १८ । मं० २४,२५ ॥

The first *mantra* means 1, 3, 5, 7, etc. It also means 1, 3^2, 5^2, 7^2, ...The second *mantra* means 4, 8, 12, 16 etc. For several such interpretations, refer to [3]. However, *Svāmī Dayānand* [17] says that several techniques of arithmetic may be derived from these *mantras*, and similarly fundamentals of algebra and geometry may be obtained from *Vedas*.

The following *mantra* from *Rig Veda* (see [17], p. 212) or [3, p. 326] talks about measuring instrument, circumference etc. :

कासीत् प्रमा प्रतिमा किं निदानमाज्यं किमासीत् परिधिः क आसीत् ।
छन्दः किमासीत् प्रउगं किमुक्थं यद्देवा देवमयजन्त विश्वे ॥४ ॥
ऋ० अ० ८ । अ० ७ । व० १८ । मं० ३ ॥

References of circle and its circumference, triangle and polygons are available in *Rig Veda* (see [3], pages 326-327).

TRITAH

Surprisingly Sanskrit name of π is *Trita*. In *Atharva Veda*, the ratio of circumference of a circle to its diameter is called *trita*, which is approximately 3:1 (see [3], p. 326).

According to Wendy O'Flaherty : "The riddles in the *Rig Veda* ... do not have, nor are they meant to have, answers. They are not merely rhetorical, but are designed to present one half of a Socratic dialogue through which the reader becomes aware of the inadequacy of his certain knowledge". I wish to say that neither it is, nor it can be the intention of the *mantra* to give a good approximation of *trita*. It is enough that the intended meaning of *trita* is not a rational number implicitly. There are several references in *Mahābhārata* which imply that the *trita* is 3. See [4] for a detailed analysis of *trita* or π in *Mhābhārata*, and [6] for a detailed analysis of the value of *trita* or π available in ancient Indian mathematical compositions. It may be mentioned that *Āryabhata's Āryabhatiya* (499 A.D.)., which is the earliest mathematical text that bears the name of the author, gives a good approximation of π (see also [1], [2], [7], [9], [10], [12], [23] - [25]).

VEDIC PERIOD

A section of *Vaidikas* hold *Veda* as beginningless (indeed as old as this creation, (see [17], pp. 12-37) and *Svayambhu*. However, the *savites* hold that *Vāgisā Siva* set forth the *Vedas*, the six *aṅgas* and *āgamas*. According to seventh century Tamil *Saivite saint Appar, Lord Siva* is the embodiment of the four *Vedas* and six *aṅgas* (see, for instance, [13], Chap. 6). However, following Keith, and Satya Prakash (see [15], pp. 5-6; and also Renfrew [16] and Datta Singh [2], Part I, pp. 1-2), *Vedas*, the earliest extant compositions, and the subsequent *Vedic* literature are assigned the following dates :

Vedas	3000 B.C.
Brāhmaṇas	2500 - 2000 B.C.
Baudhāyana Sūtra	800 B.C.
Āpastambha Sūtra	600 B.C.
Manava Sūtra	550 B.C.
Pānini Sūtra	500 B.C.
Kātyāyana Sulba Sūtra	200 B.C.

Two recent investigations push back the above mentioned date of *Vedas* significantly. The recent deciphering of Harappan language by Louisiana State University Professor and Cryptologist, S. Kak suggests *Vedic* period around 8000 B.C. A recent deep analysis under the title "*Mahayuga*: the great cosmic cycle and the date of the *Veda*" concludes the *Ṛg Veda* is at least as old as 7300 B.C. If we accept 8000 B.C. for *Ṛg Veda*, the other dates of *Brāhmaṇas* and *Sūlba Sūtras* should also be pushed back proportionately.

THEOREM ON RECTANGLES

Pythagoras theorem (also called *Baudhāyana - Pythagoras* theorem in some recent High School text books) is available implicitly in *Brāhmanas* and explicitly in *Śūlba Sūtras* which mewans "*Sūtras of cord*". For a detailed analysis refer to Seidenberg [20] - [21] (see also [2], [11], [12], [14], [15], [24] and [25]).

Consider two sets of *mantras* from *Śatapatha Brāhmaṇa*.

Set I :

तद्य एष पूर्वाध्यों वर्षिष्ठ स्थूणाराजो भवति । तस्मात्प्राङ् प्रक्रामति त्रीन्विकमां रतच्छङ्कुं निहन्ति सोऽन्तः पातः ॥१ ॥

तस्मान्मध्यमाच्छङ्कोः । दक्षिणा पञ्चदश विक्रमान्प्रक्रामति तच्छङ्कुं निहन्ति सा दक्षिणा श्रोणिः ॥२ ॥

तस्मान्मध्यमाच्छङ्कोः । उदङ् पञ्चदश विक्रमान्प्रक्रामति तच्छङ्कुं निहन्ति सोत्तरा श्रोणिः ॥३ ॥

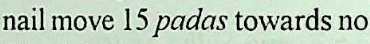
तस्मान्मध्यमाच्छङ्कोः । प्राङ् पट्त्रिꣳशतं विक्रमान्प्रक्रामति तच्छङ्कुं निहन्ति स पूर्वार्धः ॥४ ॥

तस्मान्मध्यमाच्छङ्कोः । दक्षिणा द्वादश विक्रमान्प्रक्रामति तच्छङ्कुं निहन्ति स दक्षिणोऽꣳसः ॥५ ॥

तस्मान्मध्यमाच्छङ्कोः । उदङ् द्वादश विक्रमान्प्रक्रामति तच्छङ्कुं निहन्ति स उत्तरोऽꣳस स एषा मात्रा वेदेः ॥६ ॥

अध्याय- ५ ब्राह्मण १

This means : From the tallest pole in the *s'ā lā* move 3 *padas* (foot) towards east, and fix up a nail there. Call it "*antah pā ta*". From the centre of this nail move 15 *padas* towards south, and fix up a nail there, which is called south (right) pelvic. From the centre of the nail move 15 *padas* towards north, and fix up a nail there, which is called north (left) pelvic. From the centre of the nail move 36 *padas* towards east, and fix up a nail there, which is called *Purvardha* (half-east point). From the centre of (this) nail move 12 *padas* towards north, and fix up a nail there which is called south (right) pectoral. From the centre of the nail move 12 *padas* towards north, and fix up a nail there, which is called north (left) pectoral. These are the *mā trā s* (magnitudes) of the (*Mahā*) *vedi*.



Evidently, *Mahā vedī* is *JKLM* which is an isosceles trapezoid. *P* is the starting point and *A* = *antah pā ta* where *PAE* is a line from west to east. Subsequent *mantras* give reasons for the choice of the lengths *MJ*, *LK* and *AE*. The vedi is compared with a woman whose lower part is wider than the upper part, since a woman is considered superior because of her womb which is responsible for creation (see [26], pages 429-431). Note that the *vedī* is loaded with Pythagorian triplets such as (12, 36, 39), (27, 36, 45).

Set II:

(ꣳ) स॒ व्वेद्यन्तात् । (त्ष॒) ष॒ट्त्रिꣳ शत्प्रक्रमां प्रा॒चीं व्वे॒दिं व्वि॒भिमीते त्रिꣳश॒तं प॒श्चा॒त्तिर॒श्चीं च॒तुर्व्विꣳ शति पु॒रस्ता॒त्तन्नवति॒: सैषा॒नवति॒प्रक्रमा व्वे॒दिस्त॒स्याꣳ सप्त॒विधमग्निं व्वि॒दधाति ॥४ ॥

त॒दाहु: । क॒थमेष॒ सप्त॒व्विध ऽएत॒या व्वे॒द्या स॒म्पद्यतऽइति द॒श वा॒ऽइमे पु॒रुषे प्राणा॒श्चत्वार्स॒ङ्गान्यात्मा॒ पञ्चदश॒ ऽएवं॒ द्विती॒यऽएवं॒ तृ॒तीये पट्सु॒ पु॒रुषेषु नवतिर॒थैक: पु॒रुषो॒ऽत्येति पा॒ङ्क्तो वै पु॒रुषो लो॒म त्व॒ङ्मा ꣳ सम॒स्थि मज्जा पा॒ङ्क्तोऽइयं व्वे॒दिश्चतस्त्रो दि॒श ऽआत्मा॒ पञ्च॒म्येव॒मेष॒ सप्त॒विध ऽएत॒मा व्वे॒द्या स॒म्पद्यते ॥५ ॥

These *mantras* also describe the *Mahā vedī JKLM* (cf. [18], p. 37) but there is no mention of pole P (and hence of PA). However, there is one significant aspect concerning the measurements. In fact, it says that this is 90 *padas vedi*, i.e., a *vedi* of 90 units/feet. Note that $JM + AE + KL = 30 + 36 + 24 = 90$ and $JL + MK = 45 + 45 = 90$. Note also the triangle BLJ is a right triangle and $27^2 + 36^2 = 45^2$, i.e., $JB^2 + BL^2 = JL^2$. Seidenberg [21, p. 106] has rightly said confidently that : "I therefore regard it as certain that the *Śatapatha Brahmaṇa* knows the Theorem (of Pythagoras)". He however has not mentioned the 90-*pada* - *vedi* aspect.

SAVIŚESA

Śulbas, the manual for construction of altars have different approximation for √2. This suggests an understanding of difference between raional and irrational numbers, since *Vedic* priests knew that their (approximate) value of √2 was not exact (see [8, p. 43] and [20, p. 328]). In *Śulbas*, if one wishes to turn a circle into a square (see, for instance, Baudhāyana *Śulbas* I.60; *Āpastamba Śulbas* III.9; *Kātyāyana Śulbas* III.14), then one comes across a notable contribution of *Sulbas*, viz., a rational approximation of √2, which is generally termed as *Savis'esa*. The *Śulbas* give

(V) $2 = 1 + 1/3 + 1/3.4 - 1/3.4.34$

It may be mentioned that the commentator *Rāma* of *Naimisa* (near Lucknow) gave the following improved approximation (see Kapur [9, p. 85] and Khadilkar [11, p. 83]).

(R) $\sqrt{2} = 1+1/3 + 1\backslash 3.4 - 1/3.4.34 - 1/3.4.34.33 + 1/3.4.34.34$

Several proofs of (V) have been suggested by G. Thibout (cf. [14] and S.D. Khadilkar [11] (see also [9, p. 98]). Simple arithmetical and algebraic proofs may be given to (V) and (R). The latter may further be improved as follows :

(S) $\sqrt{2} = 1+1/3 + 1/3.4 - 1/3.4.3.4 - 1/3.4.34.33 + 1/3.4.34.34$
$+ 1/3.4.34.33.34.36$
(Herein 36 may also be replaced by 35)

The approximation (V), (R) and (S) are correct to 5, 7 and 8 places of decimals respectively.

The traditional Indian or *Vedic* mathematics from *Āryabhata* (476 A.D.) to *Mādhava* (1340 - 1425 A.D.) embodies many surprising and beautiful concepts and results in algebra, geometry, trigonometry, mensuration, theory of numbers, the basic concept of limit, calculus (differential and integral both) and Taylor's series for *sines* and *cosines* upto second powers. However, most studies of Indian Mathematics have unfortunately (in fact unfortunate to historians and scientific community) not been able to see its contributions to the scientific methods. Thanks to the West that its modern savants were the first to bring an appreciation of a portion of huge Indian mathematical and astronomical compositions and the origin of geometry into the notice of the scientific community. I trust there is still a lot to be investigated, and hope that the era of neglect is perhaps over.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 83-90

STUDIES ON POLLUTION REDUCTION POTENTIAL OF EICHHORNIACRASSIPES GROWN IN INDUSTRIAL EFFLUENT

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(Received 24-12-1991)

ABSTRACT

The present investigation was conducted to find out the pollution reduction efficiency of *Eichhornia crassipes* grown in industrial wasts of Indian drug pharmaceutical limited, Virbhadra. Results obtained during this investigation showed that plant growth was healthy, leaf numbers were increased four times within 30 days and was better than that of the plants grown in tap water (control IInd). There was considerable increase in D.O. level i.e. 1.5 (mg/1t) to 3.75 (mg.It) and BOD value was decreased i.e. 240.6 to 175.0 (mg/It). But at the same time this value was reversed where *Eichhornia* plants were grown in tap water i.e. control IInd. A declined trend was recorded in other parameters such as hardness, alkalinity and chloride etc.

Keywords : *Eichhornia*. Pollution, D.O. BOD effluent.

INTRODUCTION

It is a well known fact that aquatic bodies are more susceptable for contamination due to various type of waste discharged into these. These waste includes industrial effluent, sewage and domestic water etc. Raw sewage contains various organic and inorganic materials such as cellulose, carbohydrates, fats protein urea and amines etc. Direct discharged of these polluted water often creat health problems in densly populated area of the country. Large number of waste are being discharged into aquatic bodies. These aquatic bodies contain smaller or larger number of weed dependig on nature of aquatic body. Generally aquatic weeds are still regarded by majority as "menace" and nuisance because they are not yet aware of the great potential and economic values of these profusely growing uncontrollable plants.

Eichhornia crassipes is one of the most abundantly fast growing weed in tropical and subtropical parts of the world. This fast growing troublesome weed is a good source of pulp, paper and human food. Mazid (6) reported some tribal in Bangladesh eat the flowers, young peduncle, and young petiols of this plant.

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Aquatic weeds are good source of protein as such and these may be substitute for the diet harvivorus live stock. *Water-hyacinthis* also consumed by cattle but for good consuption it has to be mixed with right proportion of other feed i.e. 30%. Cattle feed consisting equal proportion of this weed with 5% molases supported good growth of cattle (Mazid (6). This contains protein from 9.14 to 14.37/dry weight. Mazid (5) reported that 672.00, 103.74 and 84.64 lb protein is produced monthly per acre by *E. crassipes, Salvinia* sps and *Azolla pinnata* respectively in Bangla desh.

Water hyacinth can also be used as manure. Compost prepared from aquatic weeds are more effective than cow dung compost for yield in several crops. Twenty two, 25.5 and 7.5 percent more yield was gained by using compost of *E. crassipes*, than cow dung manure in Rice. Onion and Turnip respectively Mazid (6). Besides these they are used in Sudan for bio-gas production. According to NASA. Water-hyacinth harvested from one hectare will produce more than 70,000 m3 of Bio-gas.

The major role of these aquatic weeds in aquatic body to reduce the pollutants. It has been reported by National Academy of Sciences U.S.A. that some aquatic weed can scavenage inorganic and some organic compounds from water. These weeds are capable of absorbing and incorporating the dissolved materials into their own structure, Reddy and Debusk (9) reported that eight aquatic macrophytes including *E. crassipes* can remove the N and P from nutrient enriched water.

Water hyacinth is a natural resource to reduce pollution load from waste water and can be utilized for treatment of waste water. Important contribution in these lines has been made by (Schultze, (1966), Scheffield, (11); Sinha and Sinha, (12); Miner, (7); Rao et al. (8). Dunigon et al. (4); Walverton, (14); Walverton and Donald, (13)

Like other aquatic bodies, Ganga also receives various kinds of pollutants throughout its course. Albiet water pollution board in India was framed in 1974 but due to several problems it could not be so effective.

Keeping in view of the economy of the country and nutritional value as well as agricultural importance of the aquatic weed it was thought to make a preliminary study on pollution reduction potential of *E. crassipes* of larger leaves and their growth on industrial effluent under Indian environment.

MATERIALS AND METHODS

Plants of *Eichhornia crassipes* of larger leaves (Water hyacinth) used in the present investigation were collected from the local ponds. Industrial effluent was collected from Indian drug pharmaceutical limited (IDPL), a drug industry, located at Virbhadra Rishikesh, India. Fourty lts. effluent was filled in each cemented tanks. Pure eflluent (without dilution) was used for the experiment. Controls were of two types, control first was pure effluent without plants and control second was tap water with *Eichhornia* plants. Three replicate were used for each set.

Four plants of *Water hyacinth* were kept in each tank at begning of the experiment. Each plant contained seven leaves. Various physico chemical and biological characters of the effluent were evaluated *at* initial stage of the experiment. Analysis of the effluent was done as suggested by APHA [1]. A regular analysis of the effluent and observation on growth of the plants was made at interval of 15 days for the period of one month. The experiment was reconducted to confirm the findings.

RESULTS AND DISCUSSION

Results obtained during this investigation are given in the Table. In general there was no change in physico chemical characterestics of first control (Pure effluent without plants). However negligible reduction in alkalinity, hardness and BOD was recorded. Markedly reduction in various parameters was noted in pure effluent where water hyacinth plants were grown. Enhancement of dissolved oxygen from 1.5 to 3.75 (mg/lt) was found after 30 days interval. Profuse growth of the plants was found in pure effluent siamilar to the plants grown in tap water. Number of leaves were increased four times after one month. (Table).

It is very much obvious that *Eichhornia crassipes* of larger leaves is capable to reduce the polutants from water as it is evident with enhacement of dissolved oxygen in the effluent (Table). Albeit BOD was decreased gradually and dissolved oxygen was increased in the effluent but BOD was slightly increased in second control. Sixty percent reduction in BOD and suspended particles was recorded by Walverton etal. [14] when sewage was passed from wsater hyacinth plants. Further Walverton and Donald [13] reported that one acre of dens growth of waterhyacinth can treat waste water generated from 2000 people. Dinges [3], a scientist of NASA, demonstrated that this weed based filtration system can function effectively and such treatment plants for waste from municipal are under trial operation.

Increased in BOD in second control (Tap water with plant) is similar with findings of Macvea and Boyd [2]. They reported that water hyacinth decreased dissolved oxygen when it is grown in clean water. This is a peculiar feature of this plant that it reduces pollution load when grown in waste water but creates pollution when grown in clean water.

Four times increse of leaves after 30 days on effluent showes its rapid and profuse growth. This indicates that water hyacinth can grown well in effluent of such industries.

With the above findings and researches done by workers, on utility of this weed for waste treatment and in agriculture, it is desirable that this can be clultivated on stored waste water. After a definite time weed should be harvested regularly for compost preparation and cattle feed Mazid [5]. This will help in multiways at one time.

Table-IV : Effect of Eichhornia Plants on I.D.P.L. effluent.

SI. No.	Experimental tanks	No of days	No of Plants	No of Leaves	pH	Chloride	Hard-ness	Alkali-nity	Conduc-tivity	D.O.	B.O.D.
1.	Effluents 100% with plants	Q	4.00	7.00	6.20	15.00	95.00	89.00	0.230	1.50	240.60
C 1 2.	Effluents 100%	0	-	-	6.20	15.00	95.00	89.00	0.230	1.50	240.60
C11 3.	Tapwter with plants	0	4.00	7.00	7.01	10.00	175.00	162.00	0.218	8.00	0.88
1.	Effluents 100% with plants	15	4.00	16.50	5.90	5.00	70.00	80.00	0.235	2.30	210.00
C1. 2.	Effluents 100%	15	-	-	6.00	15.00	95.00	84.00	0.228	1.00	238.00
C11.3.	Tapwater with plants	15	4.00	16.50	6.80	5.00	150.00	153.00	0.220	9.80	1.40
1.	Effluents 100% with plants	30	4.00	31.50	5.80	5.00	70.00	75.00	0.230	3.75	175.00
CI 2.	Effluents 100%	30	-	-	6.00	15.00	95.00	84.00	0.223	0.40	232.00
CII 3.	Tapwater with plants	30	4.00	30.60	6.70	-	150.00	148.00	-	10.40	2.50

Control Ist (Pure without effluent plants)
Control IInd (Tap water with plant)

Developing countries like India should use such kinds of weeds (biological means) for treatment of waste water in the pattern of NASA. This would require proper weed management. A large number of aquatic weeds including waterhyacinth can be tested for treatment of municipal waste and compost preparation as well as bio gas production etc.

ACKNOWLEDGEMENT

Authors record their sincere thanks to the university authorities for their constant encouragement and facilities provided.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp. 91-94

A NOTE ON THE MODULAR SEMI-GROUP RING OF A FINITE IDEMPOTENT SEMIGROUP

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(Received 31.12.1991)

In [1] Johnson has proved when G is a group of prime power order p and K a field having only p-elements then G^*, the mod-p-envelope of G has a group structure. In this note we prove for an idempotent semi-group S of finite order having an identity I and K a field of characteristic p, S^* the mod p-envelope of S has a monoid structure, with non-trivial idempotents in it. For more properties about modular group rings please refer [1] and [2]. Just we define a semi-group ring of a semi-group S over a ring R analogous to group rings as follows:

Definition 1.

Let R be a ring (not necessarily commutative and not necessarily with a unity) and let S be a semi-group. We define the semi-group ring $R[S]$, to be the ring consisting of all finite formal sums $\Sigma\alpha_i x_i$ ($\alpha_i \in R$; $x_i \in S$) with the obvious definition of addition and with multiplication induced by the given multiplication in R and S according to the rule.$(\Sigma\alpha_i x_i)(\Sigma\beta_j Y_j) = \Sigma(\alpha_i\beta_i) x_i y_j$ (α_i, $\beta_j \in R$; x_i, $y_j \in S$). We take $(\Sigma\alpha_i x_i)$ to mean $\Sigma(\alpha_i)\ x_i$ ($\alpha_i \in R$; $x_i \in S$). If R has a unity I_R then we impose the further condition $I_R x = x$ for all $x \in S$. In particular $R[S]$ can be viewed as a free left R-module. When R is a commutative field and S a monoid $R[S]$ is a semi- group algebra. Throughout this paper by S we mean a commutative idempotent semi-group under multiplication with multiplicative identity 1.

Example 1.

Let $K = (0, 1)$ and $S = \{1, g_1, g_2 \mid g_1{}^2 = g_1, g_2{}^2 = g_2, g_1g_2 = g_2g_1\}$ be a semigroup. KS the semigroup ring of S over K is given by $KS = \{0, 1, g_1, g_2, g_1g_2, 1+g_1, 1+g_2, 1+g_1 g_2, g_I + g_2, g_I + g_1 g_2, g_2 + g_1 g_2, 1+ g_1 + g_2, 1+ g_1 + g_1 g_2, 1+ g_2 + g_1 g_2, g_1 + g_2 + g_1 g_2. 1 + g_1 + g_2 + g_1 g_2\}$. $S^* = I+U$ *where* $U = \{\Sigma\alpha_i g_i \mid \Sigma\alpha_i = 0\}$ ($\alpha_i \in K$, $g_i \in S$); $S^* = I+U = I+ \{g_I + g_2, 1+g_1, 1+g_2, 1+g_1 g_2, g_I + g_1 g_2, g_2 + g_1 g_2, 1 + g_1 + g_2 + g_1 g_2, 0\} = \{g_1, g_2, g_1g_2, 1 + g_1 + g_2, 1, 1+g_1 + g_1 g_2, 1 + g_2 + g_1 g_2, g_1 + g_2 + g_1 g_2\}$ clearly S^* *is* an idempotent semi-group with identity and order of S^* is 8.

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Definition 2.
Let S be a semi-group with identity and K a field of characteristic p. KS the semi-group ring of S over K, then S^*, the mod p-envelope of S is $1+U$ where $U = \{ \Sigma\alpha_i s_i \mid \Sigma\alpha_i = 0,\ \alpha_i \in K;\ S_i \in S \}$.

Proposition 3.
Le $K = (0,1)$ be the field of integers modulo 2 and $S = \{1, g_i \mid g_i^2 = g_i\ g_i g_j = g_j g_i\}$ be an idempotent semigroup of order n. Then S^* is a monoid in which every element is an idempotent and order of S^* is 2^{n-1}.

Proof.
$S^* = 1+U$ where $U = \{ \Sigma\alpha_i s_i \mid \Sigma\alpha_i = 0 \}$.Here $U = \{0,$ elements taken two at a time, elements taken four at a time,..., elements taken n at a time if n is even or elements taken n-1 at a time if n is odd). Thus order of S^* is 2^{n-1}. This is easy to verify $1 \in S^*$ and every element of S^* is an idempotent as characteristic of K is two.

Example 2.
Let $K = (0, 1,2)$ be the field of integers modulo 3 and $S = \{1,\ g_1 g_2 \mid g_1^2 = g_1,\ g_2^2 = g_2,\ g_1 g_2 = g_2 g_1\}$ Then S^* is not an idempotent semigroup but contains identity and idempotents.For $S^* = 1+ U = 1+ \{ g_1 + g_2 + g_1 g_2,\ g_1 + 1 + g_1 g_2,\ g_1 + g_2 + 1,\ 1 + g_2 + g_1 g_2,\ 2 + 2g_1 + 2g_1 g_2,\ 2g_1 + 2g_2 + 2g_1 g_2,\ 2 + 2g_2 + 2g_1 g_2,\ 2 + 2g_1 + 2g_2,\ g_1 + g_2 + 2 + 2g_1 g_2,\ 2g_1 + 2g_2, + g_1 g_2 + 1,\ 2g_1 + 2 + g_2 + g_1 g_2,\ 2g_1 + 2g_1 g_2 + g_2 + 1,\ 2g_1 g_2 + 2g_2 + g_1 + 1,\ 2g_2 + 2 + g_1 g_2 + g_1,\ 2g_1 + g_2,\ 2g_1 + 1,\ 2g_1 + g_1 g_2,\ g_1 + 2g_2,\ g_1 + 2g_1 g_2,\ g_1 + 2,\ g_2 + 2,\ 2 + g_1 g_2,\ 2g_1 g_2 + 1,\ 2g_1 g_2 + g_2, g_1 g_2 + 2g_2,\ 2g_1 + 1,\ 0\}$ Now S^*contains idempotents and not every element of S^* is an idempotent. $1 \in S^*$ with order of $S^* = 27$.

Proposition 4.
Let $Z_p = (0,\ 1, \ldots, p\text{-}1)$ be the field of integers mod-p, p a prime and S be a commutative idempotent semi group of order n with identity 1. Then the following conditions are true in S^*.

(1) S^*, the mod p envelope of S has p^{n-1} elements
(2) $S \underline{C} S^*$
(3) S^* has no zero divisors.

Proof.
Since $S^* = 1+U$ where $U = \{ 0,\ \Sigma\alpha_i s_i \mid \Sigma\alpha_i = 0,\ \alpha_i \in K$ and $s_i \in S\}$ we have order

of S^* to be p^{n-1}. Clearly SCS^*

If $\alpha \in S^*$ be such that there exists a $\beta \in S^*$ with $\alpha\beta = 0$ then we will arrive at a contradiction. Let $\alpha = \Sigma \alpha_i s_i$ and $\beta = \Sigma \beta_j s_j$ with $\Sigma\alpha_i = \Sigma\beta_j = 1$ $\alpha\beta = \Sigma\alpha_i s_i \; \Sigma\beta_j s_j = \Sigma(\alpha_i\beta_i)\, s_i s_j$ if $\alpha\beta = 0$ we must have $\Sigma\alpha_i\beta_j = 0$. Obviously SCS^* as every element of S is contained in S^* (Since $(p-1) + s_i \in U$ for all $s_i \in S$ so $1 + (p-1) + s_i \in S^*$ ie $s_i \in S^*$), S^* has no zero divisors. For if S^* contains zero divisors say α, β such that $\alpha.\beta = 0$; then we must have $\Sigma\alpha_i \; \Sigma\beta_j = \Sigma\alpha_i\beta_j = 0$ (where $\alpha = \Sigma\alpha_i s_i$ and $\beta = \Sigma\beta_i s_i$) which is impossible as $\Sigma\alpha_i = 1$ and $\Sigma\beta_j = 1$, we have $\Sigma\alpha_i\beta_j = 1$ as both α and β are in S^*. Thus S^* has no zero divisors.

Example 4.

Let $K = (0,1)$ and $S = < s, 1 \mid s^3 = s >$. S^* is a semi group having an idempotent and identity. For $KS = \{0, 1, s, s^2, 1+s, 1+s^2, 1+s+s^2, s+s^2\}$, Now $S^* = 1 + U = 1 + \{0, 1+s, 1+s^2, s+s^2\} = \{1, s, s^2, 1+s+s^2\}$ Now $s^2 \in S^*$ is an idempotent of S^*.

Theorem 5.

Let $K = (0, 1)$ and S be a semigroup with 1 and an element s such that $s^p = s$ where p is any integer. Then S^* is a semigroup with 1and order of S^* is 2^{p-1}.

Proof obvious

Example 5.

Let $K = (0,1,2)$ and $S = \{1, s \mid s^3 = s)$. The $KS = \{0, 1, 2, s, s^2, 2s, 2s^2, 1+s, 1+s^2, 1+s+s^2, s+s^2, 2+2s, 2+2s^2, 2+2s+2s^2, 2s+2s^2, 1+2s+2s^2, 2+s+2s^2, 2+s^2+2s, 2+s+s^2, 1+s^2+2s, 1+2s^2+s, s+2s^2, 1+2s^2, 1+2s, 2+s^2, 2+s^2, 2+s, 2s+s^2\}$. $S^* = 1 + U = 1 + \{1+s+s^2, 2+2s+2s^2, 2+s, 0, 2s+s^2, 2+s^2, 1+2s^2, 1+2s, s+2s^2\}$. $S^* = \{1, 2+s+s^2, 2s+2s^2, s, s^2, 2+2s^2, 1+2s+s^2\}$ S^* is a semigroup with one has elements of order two and order of S^* is 9. Thus

Theorem 6. Let $K = (0, 1, \dots, p-1)$; p a prime and $S = \{s, 1 \mid s^p = s\}$. Then S^* is a semigroup of order p^{p-1} with identity 1.

Proof Obvious.

We are not able to say anything when S is a general semigroup.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp. 95-104

STRESS DISTRIBUTION IN A NON - HOMOGENEOUS CYLINDRICALLY ANISOTROPIC ELASTIC CYLINDER

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(Received 03.01.1992)

ABSTRACT

This paper deals with the determination of the radial displacement and relevant stresses in a cylinder on the basis of three dimensional linear theory of elasticity. The material of the cylinder is orthotropic with cylindrical anisotropy and, in addition, is continuously non-homogeneous, with mechanical properties varying along the radius. The variations of stresses along the radius have been shown graphically.

Mathematical Subject Classification (1991) : 76

Keywords: Orthotropic material, Cylindrical anisotropy, Elastic parameter, Nonhomogeneous material, Hypergeometric function.

INTRODUCTION

The solution for radial deformation and corresponding stresses in a cylindrical shell of homogeneous isotropic elastic media under the action of internal and external pressure was obtained by Lame' as quoted in Love [5].

Since last few decades many investigators have been paying attentions to problems in elasticity in which the material is no longer homogeneous. Due to complications in the governing differential equations these problems are sometimes solved by numerical methods. Grief and Chou [3] have considered such a problem. They have solved the vibration problem of an anisotropic non-homogeneous cylindrical shell without considering any longitudinal extension. Roy [7] has found the radial deformation and stresses in a cylindrically anisotropic non-homogeneous material under the influence of normal pressure on both the boundaries, allowing uniform longitudinal extension. In our present investigation we have considered the problem of stress distribution in an elastic

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cylinder of non-homogeneous material. The medium is assumed to be such that the elastic coefficients at any point vary exponentially with an arbitrary power of the radial distance of the point considered. Due to this assumption the governing differential equation reduces to a confluent hypergeometric equation by suitable transformation. The expression for the radial displacement and stresses in the homogeneous case have been shown to be the limiting values of the corresponding expressions of our results.

FORMULATION OF THE PROBLEM

Taking the axis of anisotropy as the z-axis and using cylindrical coordinate (r, θ, z) the stress-strain relations for cylindrically orthotropic material take the form (cf. Lekhnitskii [4], Armenakas and Ritz [2])

$$\sigma_r = c_{11}e_{rr} + c_{12}e_{\theta\theta} + c_{13}e_{zz}$$

$$\sigma_\theta = c_{21}e_{rr} + c_{22}e_{\theta\theta} + c_{23}e_{zz}$$

$$\sigma_z = c_{31}e_{rr} + c_{32}e_{\theta\theta} + c_{33}e_{zz}$$

$$\tau_{\theta z} = c_{44}e_{\theta z}, \quad \tau_{rz} = c_{55}e_{rz}, \quad \tau_{r\theta} = c_{66}e_{r\theta} \tag{1}$$

where $c_{ij} = c_{ji}$, $(i,j = 1,2,3,)$

The strain components in terms of the displacement (u,o,w) are

$$e_{rr} = \partial u/\partial r, \; e_{\theta\theta} = u/r, \; e_{zz} = \partial w/\partial z$$

$$e_{rz} = \partial u/\partial z + \partial w/\partial r, \; e_{r\theta} = e_{\theta z} = 0 \tag{2}$$

Let the nonhomogeneity of the material be characterised by the variations of the elastic parameters c_{ij} according to the following law

$$c_{ij} = \lambda_{ij}\, e^{-kr^m}, \; k > 0 \tag{3}$$

Where $\lambda_{ij}'s$ are the prescribed parameters of the material concerned and $m(\neq 0)$ is a real number. The case $m = 0$ corresponds to the homogeneous medium.

For the case

$$w = ez \tag{4}$$

where e is a constant and $\partial u/\partial z = 0$, we have a state of plane-strain with a uniform longitudinal extension superposed.

In view of (4) $\tau_{rz} = 0$ and the stress equation of equilibrium in the absence of the body forces reduce to

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0 \tag{5}$$

Using equations (2), (3) and (4) in (1) we get the stresses in terms of the displacement, then substitution of the stresses in (5) yields the equation of equilibrium as

$$r^2 \frac{d^2 u}{dr^2} + (1 - mkr^m) r \frac{du}{dr} - \left(\frac{\lambda_{22} + \lambda_{12} kmr^m}{\lambda_{11}} \right) u$$

$$= \frac{\lambda_{13}}{\lambda_{11}} emkr^{m+1} - \left(\frac{\lambda_{13} - \lambda_{23}}{\lambda_{11}} \right) er \tag{6}$$

If we assume that the cylinder $a \le r \le b$ is under the action of internal and external pressures, the boundary conditions may be put as

$$\sigma_r = -p_o \quad \text{at} \quad r = a$$

$$\sigma_r = -p_1 \quad \text{at} \quad r = b \tag{7}$$

Our problem is to solve equation (6) satisfying (7).

Solution of the problem

In view of the transformation

$$z = kr^m, \; u = z^n V, \; n = s/m \tag{8}$$

the equation (6) reduces to

$$z \frac{d^2 V}{dz^2} + (\gamma - z) \frac{dV}{dz} - \alpha V = z^q (\xi z + \eta) \tag{9}$$

where

$$\alpha = (s + \lambda_{12}/\lambda_{11})p$$

$$\gamma = 2n+1$$

$$q = p - n - 1$$

$$s = \sqrt{(\lambda_{22}/\lambda_{11})} = mn$$

$$p = 1/m$$

$$\xi = \lambda_{13}/\lambda_{11}\ epk^{-p}$$

$$\eta = (\lambda_{23} - \lambda_{13})/\lambda_{11}\ ep^2 k^{-p}$$

The complementary equation

$$z\frac{d^2V}{dz^2} + (\gamma - z)\frac{dV}{dz^2} - \alpha V = 0 \tag{10}$$

of the differential equation (9) is a confluent hypergeometric equation whose solution is known (Abramowitz and Stegun [1]) for integral as well as for non integral values of γ, being given by

$$V = AM(\alpha\,;\gamma\,;z) + B\,\phi(z) \tag{11}$$

In (11) $M(\alpha\,;\gamma\,;z)$ is the confluent hypergeometric function such that

$$M(\alpha;\gamma;z) = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n}\frac{z^n}{n!} \tag{12}$$

Where $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)$ with $(\alpha)_o = 1$.

The series on the right hand side of (12) is known to be convergent (Rainville[6]) for all finite z unless $\gamma = 0$ or a negative integer.

The function $\varphi(z)$ in (11) depends on γ as follows :

$\varphi(z) = z^{1-\gamma}\,M(\alpha+1-\gamma\,;2-\gamma\,,z)$ when γ is not an integer

$$= \frac{(-1)^{K+1}}{K!\,\Gamma(\alpha-K)}\,[M(\alpha;\,K+1;\,z)\,\ln z$$

$$+ \sum_{r=0}^{\infty}\frac{(\alpha)_r z^r}{(k+1)_r\, r!}\,\{\psi(\alpha+r) - \psi(1+r) + \psi(1+K+r)\}]$$

$$+ \frac{(K-1)!}{\Gamma(\alpha)}\, z^{-K} M(\alpha-K;\, 1-K;\, z)_K \qquad (13)$$

When γ is an integer and $= K+1$, $(K = 0,1,2,\ldots)$, the last function being the sum to K terms. It is to be interpreted as zero when $K = 0$, and $\psi(\alpha) = \dfrac{\Gamma'(\alpha)}{\Gamma(\alpha)}$.

The particular solution of (9) is given by

$$V = Z^{\beta}\,[a_0 + a_1 z\ {}_2F_2\,(1, \alpha+\beta+1;\, \beta+2,\, \beta+\gamma+1\, ,z)]$$

$= f(z)$, say

Where

$$a_0 = \frac{\eta}{\beta(\beta+\gamma-1)}$$

$$a_1 = \frac{1}{(\beta+1)(\beta+\gamma)}\left[\xi + \frac{\eta(\alpha+\beta)}{\beta(\beta+\gamma+1)}\right]$$

$\beta = p-n$ and ${}_2F_2\,(a, b; c, d; z)$ is the generalised hypergeometric function such that the series represented by ${}_2F_2\,(a, b; c, d; z)$ is convergent for all finite z, provided, of course, that neither c nor d is zero or a negative integer.

Thus the general solution of (9) is known. Hence using (8) the displacement u is given by

$$u = z^n\,[AM\,(\alpha\,;\,\gamma\,;\,z) + B\,\phi\,(z) + f\,(z)] \qquad (14)$$

Substituting (14) in (2) we get the components of strain. The components of stress may be obtained from (1) together with (3) and (8) as

$$\sigma_r = A\, h_1(z) + B\, h_2(z) + g(z) \tag{15}$$

$$\sigma_\theta = A\, h_3(z) + B\, h_4(z) + f_1(z) \tag{16}$$

$$\sigma_z = A\, h_5(z) + B\, h_6(z) + f_2(z) \tag{17}$$

where

$$f_1(z) = [z^{n-p}\{c_4 f(z) + c_6 z f'(z)\} + e\,\lambda_{23}] \exp(-z),$$

$$f_2(z) = [z^{n-p}\{c_7 f(z) + c_9 z f'(z)\} + e\,\lambda_{33}] \exp(-z),$$

$$g(z) = [z^{n-p}\{c_1 f(z) + c_3 z f'(z)\} + e\,\lambda_{13}] \exp(-z),$$

$$h_1(z) = z^{n-p}[c_1 M(\gamma-\alpha; \gamma; -z) + c_2 z M(\gamma-\alpha; \gamma+1; -z)],$$

$$h_2(z) = z^{n-p}[c_1 \phi(z) + c_3 z \phi'(z)] \exp(-z),$$

$$h_3(z) = z^{n-p}[c_4 M(\gamma-\alpha; \gamma; -z) + c_5 z M(\gamma-\alpha; \gamma+1; -z)],$$

$$h_4(z) = z^{n-p}[c_4 \phi(z) + c_6 z \phi'(z)] \exp(-z),$$

$$h_5(z) = z^{n-p}[c_7 M(\gamma-\alpha; \gamma; -z) + c_8 z M(\gamma-\alpha; \gamma+1; -z)],$$

$$h_6(z) = z^{n-p}[c_7 \phi(z) + c_9 z \phi'(z)] \exp(-z),$$

$$c_1 = k^p (mn\lambda_{11} + \lambda_{12}),$$

$$c_2 = \frac{\alpha}{\gamma}\, m\lambda_{11}\, k^p,$$

$$c_3 = m\lambda_{11}\, k^p,$$

$$c_4 = k^p (mn\lambda_{12} + \lambda_{22}),$$

$$c_5 = \frac{\alpha}{\gamma}\, m\lambda_{12}\, k^p.$$

$$c_6 = m\lambda_{12}\, k^p,$$

$$c_7 = k^p\,(mn\lambda_{13} + \lambda_{23}),$$

$$c_8 = \frac{\alpha}{\gamma}\; m\,\lambda_{13}\; k^p,$$

$$c_9 = m\lambda_{13}\, k^p.$$

Finally the boundary conditions (7) determine A and B as

$$A = [g_2(z_2)\, h_2(z_1) - g_1(z_1) h_2(z_2)]/\Delta$$

$$B = [h_1(z_2)\, g_1(z_1) - h_1(z_1) g_2(z_2)]/\Delta \qquad (18)$$

where

$$\Delta = h_2(z_1)\, h_1(z_2) - h_1(z_1)\, h_2(z_2)$$

$$g_1(z) = -p_0 - g(z),\; g_2(z) = -p_1 - g(z)$$

$$z_1 = ka^m,\; z_2 = kb^m$$

Substitution of (18) in (14), (15), (16) and (17) determines displacement and stresses completely.

A partial check of the results presented in this discussion has been made by making $m \to 0$ and $k \to 0+$. Thus we see from (3) that λ_{ij} are the elastic coefficients in the homogeneous case. Here we shall confine our attention only to the case in which γ is not an integer. Choosing appropriate $\phi(z)$ from (13) and letting m $\to$ 0 and k $\to$ 0 + we get the displacement and stresses as

$$u = \frac{ab}{(s\lambda_{11} + \lambda_{12})}\left[\frac{q_0}{b}\left(\frac{r}{b}\right)^s - \frac{q_1}{a}\left(\frac{r}{a}\right)^s\right] \Big/ \left[\left(\frac{b}{a}\right)^s - \left(\frac{a}{b}\right)^s\right]$$

$$+ \frac{ab}{(\lambda_{12} - s\lambda_{11})}\left[\frac{q_1}{a}\left(\frac{a}{r}\right)^s - \frac{q_0}{b}\left(\frac{b}{r}\right)^s\right] \Big/ \left[\left(\frac{b}{a}\right)^s - \left(\frac{a}{b}\right)^s\right]$$

$$+ \frac{(\lambda_{23} - \lambda_{13})er}{\lambda_{11} - \lambda_{12}}. \tag{19}$$

$$\sigma_r = \frac{\dfrac{ab}{r}\left\{\left[\dfrac{q_0}{b}\left(\dfrac{r}{b}\right)^s - \dfrac{q_1}{a}\left(\dfrac{r}{a}\right)^s\right] + \left[\dfrac{q_1}{a}\left(\dfrac{a}{r}\right)^s - \dfrac{q_0}{b}\left(\dfrac{b}{r}\right)^s\right]\right\}}{\left[\left(\dfrac{b}{a}\right)^s - \left(\dfrac{a}{b}\right)^s\right]}$$

$$+ \frac{(\lambda_{23} - \lambda_{13})}{\lambda_{11} - \lambda_{22}}(\lambda_{11} + \lambda_{12})e + \lambda_{13}e \tag{20}$$

$$\sigma_\theta = \frac{\dfrac{ab}{r}\left\{\begin{aligned}&\left(\dfrac{s\lambda_{12} + \lambda_{22}}{s\lambda_{11} + \lambda_{12}}\right)\left[\dfrac{q_0}{b}\left(\dfrac{r}{b}\right)^s - \dfrac{q_1}{a}\left(\dfrac{r}{a}\right)^s\right]\\ &+ \left(\dfrac{\lambda_{22} - s\lambda_{12}}{s\lambda_{12} - \lambda_{11}}\right)\left[\dfrac{q_1}{a}\left(\dfrac{a}{r}\right)^s - \dfrac{q_0}{b}\left(\dfrac{b}{r}\right)^s\right]\end{aligned}\right\}}{\left[\left(\dfrac{b}{a}\right)^s - \left(\dfrac{a}{b}\right)^s\right]}$$

$$+ \left(\frac{\lambda_{23} - \lambda_{13}}{\lambda_{11} - \lambda_{22}}\right)(\lambda_{12} + \lambda_{22})e + \lambda_{23}e \tag{21}$$

$$\sigma_z = \frac{\dfrac{ab}{r}\left\{\begin{aligned}&\left(\dfrac{s\lambda_{13} + \lambda_{23}}{s\lambda_{11} + \lambda_{12}}\right)\left[\dfrac{q_0}{b}\left(\dfrac{r}{b}\right)^s - \dfrac{q_1}{a}\left(\dfrac{r}{a}\right)^s\right]\\ &+ \left(\dfrac{\lambda_{23} - s\lambda_{13}}{\lambda_{12} - s\lambda_{11}}\right)\left[\dfrac{q_1}{a}\left(\dfrac{a}{r}\right)^s - \dfrac{q_0}{b}\left(\dfrac{b}{r}\right)^s\right]\end{aligned}\right\}}{\left[\left(\dfrac{b}{a}\right)^s - \left(\dfrac{a}{b}\right)^s\right]}$$

$$+ \left(\frac{\lambda_{23} - \lambda_{13}}{\lambda_{11} - \lambda_{22}}\right)(\lambda_{13} + \lambda_{23})e + \lambda_{33}e \tag{22}$$

$$q_0 = p_0 + \frac{\lambda_{23} - \lambda_{13}}{\lambda_{11} - \lambda_{22}} (\lambda_{11} + \lambda_{12})e + \lambda_{13}e$$

$$q_1 = p_1 + \frac{\lambda_{23} - \lambda_{13}}{\lambda_{11} - \lambda_{22}} (\lambda_{11} + \lambda_{12})e + \lambda_{13}e$$

These results are in complete agreement with the results of the homogeneous case.

If the material of the structure be isotropic and homogeneous the material constants λ_{ij} become

$$\lambda_{11} = \lambda_{22} = \lambda_{33} = \lambda + 2\mu\ ,\ \lambda_{12} = \lambda_{13} = \lambda_{23} = \lambda$$

In this case $s = 1$ and the displacement and the stress components have the same expressions as those given in Love [5].

Numerical Results

Numerical computations of stresses have been made by taking

$$\begin{array}{llll} \lambda_{11} = 40.75 & \lambda_{22} = 50.61 & \lambda_{33} = 42.63 & \lambda_{44} = 15.66 \\ \lambda_{55} = 19.14 & \lambda_{66} = 19.00 & \lambda_{23} = 12.76 & \lambda_{13} = 12.18 \\ \lambda_{12} = 18.27, & & & \end{array}$$

which are the material constants for topaz. Here λ_{ij}'s are expressed in term

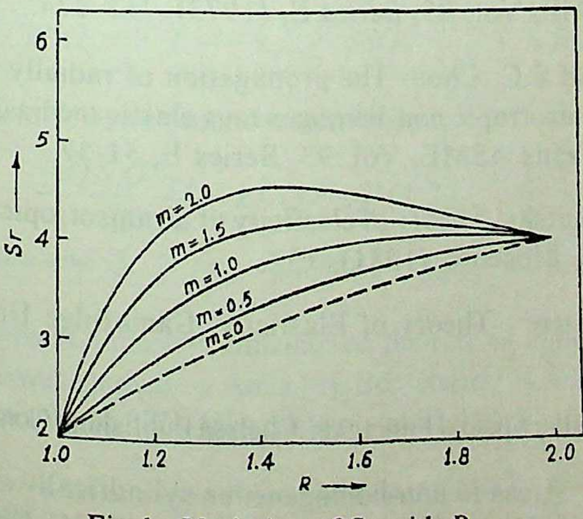

Fig. 1 : Variation of Sr with R

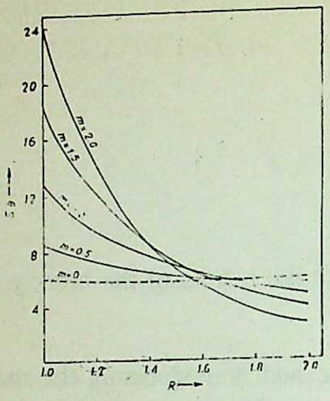

Fig.2 : Variation of S_θ with R

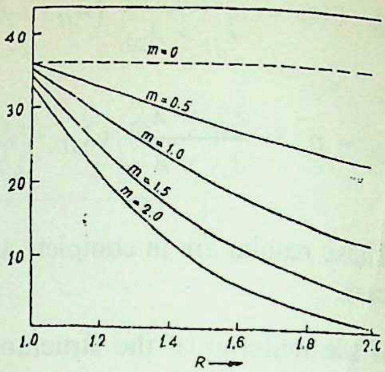

Fig.3 : Variation of S_Z with R

of a unit stress of $10^6\ lb/(in)^2$. In addition, we assume that $\delta = 2 \times 10^6\ lb(in)^2$ and write $p_0 = \delta e$, $p_1 = 2\delta e$. With this assumption, we plot the values of $S_r = -\sigma r/e$, $S_\theta = -\sigma \theta/e$ and $S_z = -\sigma z/e$, against $R = (r/a)$ in $1 \leq R \leq 2$, in Fig. 1, Fig. 2 and Fig.3 respectively. In our calculation we have considered $m = 0.0, 0.5, 1.0, 1.5, 2.0$. From the figures it is observed that the stresses change regularly with variations of m.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 105-118

A STUDY OF SHAPE OF ACTIVATED COMPLEX IN REACTION BETWEEN POTASSIUM PEROXODISULPHATE AND POTASSIUM IODIDE

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(Record 17.07.92)

ABSTRACT

The rate constants of the reaction between potassium peroxodisulphate and potassium iodide have been measured in a series of water acetone mixtures at 30± 0.1°C and d40 ± 0.1°C at various ionic strengths (μ). The radii of activated complexes for single sphere and double sphere models have been calculated from the slopes of linear plots of logarithm of rate constant at zero ionic strength (k_0) versus reciprocal of dielectric constant $(1/\varepsilon)$. Results show that the description of the activated complex could best be given by the single sphere model.

An attempt has also been made to evaluate the values of electrostatic contributions to the free energies changes $(\Delta G^{\neq}_{e.s})$.

INTRODUCTION

The reaction between peroxodisulphate and iodide ions takes place in two steps [1,2]

$$S_2O_8^{--} + I^- \longrightarrow IS_2O_8^{--} \qquad \text{slow} \ldots (1)$$

$$IS_2O_8^{---} + I^- \longrightarrow I_2 + 2SO_4^{--} \qquad \text{fast} \ldots (2)$$

This reaction follows second order kinetics:

$$\frac{-d[S_2O^{--}_8]}{dt} = k[S_2O^{--}_8]\ [I^-]z$$

where 't' is time and 'k' is rate constant.

The influence of dielectric constant on the rate of ionic reactions having similar charges was studied by Amis [3], Scatchard [4], Laidler [5], Laidler-Eyring [6], Christiansen [7], Ghaziuddin - et al. [8-10] and Fahim Uddin et

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al. [11-13]. Ghaziuddin - et al. [14] and Fahim Uddin et al. [15] also studied the reaction between peroxodisulphate-iodide ions in ethanol water and methanol water mixtures respectively and found that the activated complex is single sphere for both the media i.e. ethanol water and methanol water mixtures.

In the present work, the reaction between potassium peroxodisulphate and potassium iodide has been studied to confirm whether the activated complex consists of single sphere or double sphere model.

EXPERIMENTAL

Potassium peroxodisulphate, potassium iodide, sodium thiosulphate, potassium dichromate, starch, acetone used were of E. Merck. Double distilled water was used throughout the course of experiments.

Dielectric constants of the medium were varied by changing proportions of water acetone mixtures. The composition of water acetone mixtures *(W/W)* were 8. 08%, 12.25%, 16.51%, 20.07% and 25.32% and respective values of dielectric constants were taken from Akerlof [16-17].

The rate constants of the reaction were measured by a titrimetric method. Calculated volumes of $K_2S_2O_8$ and KI were taken in 25 ml volumetric flasks and diluted up to the mark. These two diluted solutions were kept in a thermostatic bath (Type K-33 Haake, Karlsruhe, Germany) at constant temperature for half an hour. Then these two solutions were mixed and the time was recorded. A 5 ml portion of this reaction mixture was withdrawn after each interval of 5 minutes. The mixture was chilled to freeze the reaction and titrated against a standard solution of sodium thiosulphate. 2 drops of starch solution was added as an indicator.

RESULTS AND DISCUSSIONS

The rate constants were calculated using the integrated form of the rate expression for a second order reaction with the same initial concentrations i.e.,

$$k = \frac{1}{t} \cdot \frac{x}{a(a-x)} \qquad \ldots\ldots.(4)$$

where *'a'* is the initial concentration of potassium peroxodisulphate and potassium iodide, x is the concentration of the product at time *'t'* and *(a-x)* is the remaining concentration of the reactant at time *'t'*.

Plots of *[x/a(a-x)]* against time 't' were drawn at different ionic strengths and composition of solute i.e. acetone water mixtures. A representative plot of *[x/a(a-x)]* versus *'t'* at ionic strength (μ) = 7.84×10^{-2} mil.dm^{-3}in 10% acetone water mixtures at 40°C is shown in figure - 1. From the slopes of the plots, rate constants were evaluated. The values of rate constants for the

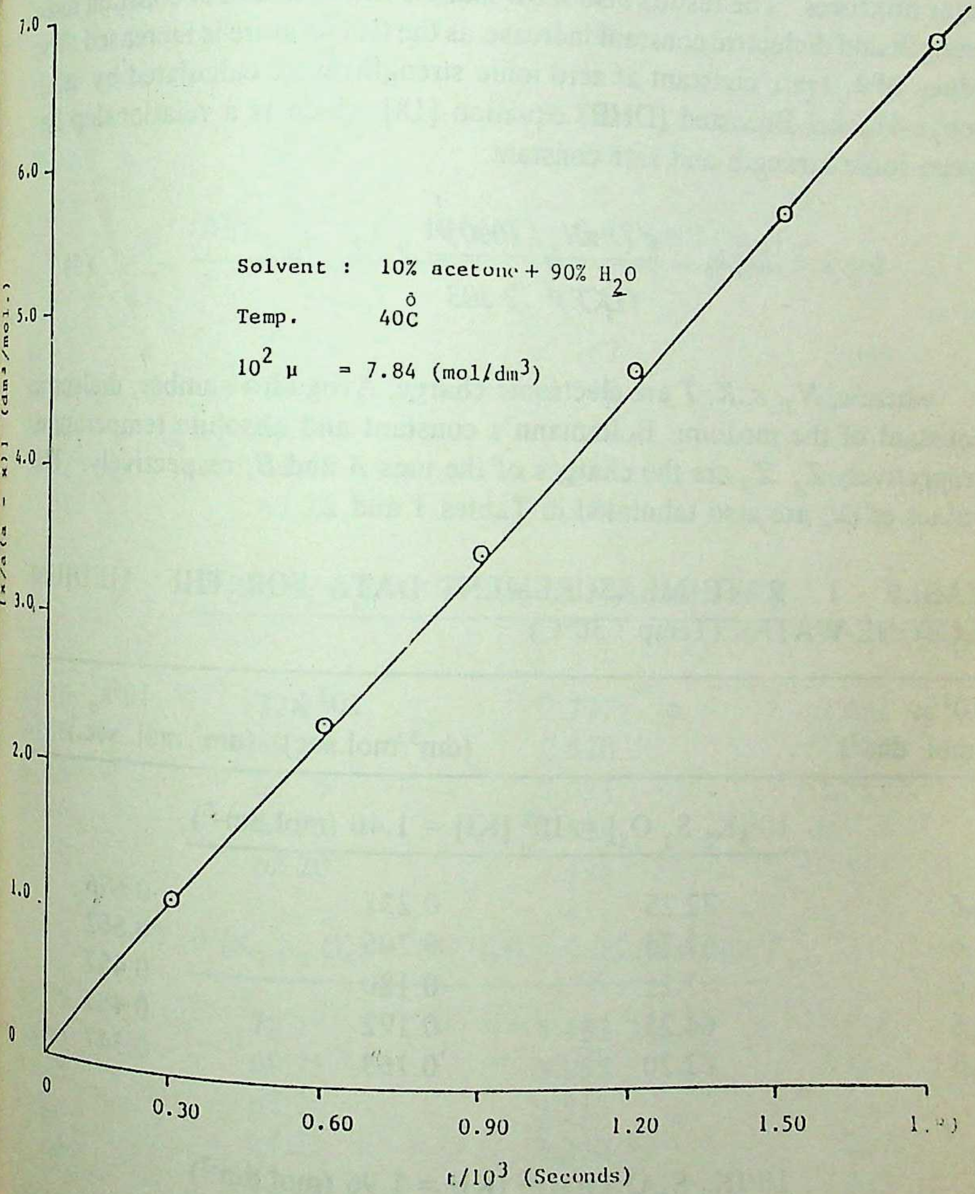

FIGURE - 1

Fig 1 : Plot of *[x/a (a-x)]* versus *'t'* at 40°C

reaction between peroxo-disulphate and iodide ions in acetone water mixtures at 30° and 40°C are summarized in Tables 1 and 2 respectively.

It was also noted that for the same value of dielectric constant, the rate constant increased with increasing ionic strength. These results are in agreement with those reported earlier [14] for the same reaction in ethanol water mixtures. The results also show that the rate constant at constant ionic strength and dielectric constant increase as the temperature is increased. The values of k_o (rate constant at zero ionic strength) were calculated by using Debye-Huckel-Bronsted (DHB) equation [18] which is a relationship between ionic strength and rate constant:

$$\log k = \log k_0 + \frac{e^3 (8\pi N_A / 1000)^{\frac{1}{2}}}{(\varepsilon KT)^{\frac{3}{2}} \,.\, 2.303} . Z_A . Z_B . \sqrt{\mu} \qquad \ldots\ldots\ldots(5)$$

where e, N_A, ε, K, T are electronic charge, Avogadro number, dielectric constant of the medium, Boltzmann's constant and absolute temperature, respectively Z_A, Z_B are the charges of the ions A and B, respectively. The values of k_o are also tabulated in Tables 1 and 2.

TABLE - 1 : RATE MEASUREMENT DATA FOR THE MEDIUM ACTONE-WATER (Temp : 30°C)

10^2 μ (mol. dm^{-3})	ε	10^3 k (dm^3/mol.sec)	$10^4 k_o$ (dm^3/mol. sec.)
	$10^2[K_2S_2O_8] = 10^2$ [KI] = 1.40 (mol.dm^{-3})		
5.6	72.25	0.231	0.676
5.6	69.75	0.205	0.562
5.6	67.25	0.180	0.457
5.6	64.25	0.172	0.403
5.6	62.20	0.163	0.347
	$10^2[K_2S_2O_8] = 10^2$ [KI] = 1.96 (mol.dm^{-3})		
7.84	72.25	0.677	1.585
7.84	69.75	0.625	1.349
7.84	67.25	0.419	0.832

7.84	64.25	0.342	0.616
7.84	62.20	0.278	0.447
$10^2[K_2\ S_2\ O_8] = 10^2\ [KI] = 2.52\ (mol.dm^{-3})$			
10.08	72.25	1.820	3.467
10.08	69.75	1.090	1.905
10.08	67.25	0.963	1.549
10.08	64.25	0.872	1.259
10.08	62.20	0.653	0.830
$10^2[K_2\ S_2\ O_8] = 10^2\ [KI] = 3.08\ (mol.dm^{-3})$			
12.32	72.25	2.167	2.089
12.32	69.75	1.318	1.950
12.32	67.25	1.042	1.380
12.32	64.25	0.983	1.148
12.32	62.20	0.876	0.849
$10^2[K_2\ S_2\ O_8] = 10^2\ [KI] = 5.04\ (mol.dm^{-3})$			
20.16	72.25	2.777	2.692
20.16	69.75	2.670	2.291
20.16	67.25	2.031	1.513
20.16	64.25	1.538	0.977
20.16	62.20	1.495	0.813
$10^2[K_2\ S_2\ O_8] = 10^2\ [KI] = 7.00\ (mol.dm^{-3})$			
28.00	72.25	5.483	3.516
28.00	69.75	3.958	2.188
28.00	67.25	3.513	1.659
28.00	64.25	3.230	1.267
28.00	62.20	2.602	0.832

TABLE - 2 RATE MEASUREMENT DATA FOR THE MEDIUM ACETONE WATER (Temp : 400C)

$10^2 \mu$ (mol. dm-3) sec.)	ε	10^3 k (dm^3/mol.sec)	$10^4 k_o$ (dm^3/mol.
$10^2[K_2\ S_2\ O_8] = 10^2\ [KI] = 1.40\ (mol.dm^{-3})$			
5.60	69.00	1.967	5.623
5.60	66.75	1.823	4.898
5.60	64.25	1.592	3.981
5.60	62.00	1.102	2.521
5.60	59.75	1.061	2.239
$10^2[K_2\ S_2\ O_8] = 10^2\ [KI] = 1.96\ (mol.dm^3)$			
7.84	69.00	3.816	8.709
7.84	66.75	3.683	7.762
7.84	64.25	2.516	4.786
7.84	62.00	1.616	2.818
7.84	59.75	1.482	2.344
$10^2[K_2\ S_2\ O_8] = 10^2\ [KI] = 2.52\ (mol./dm^3)$			
10.08	69.00	4.762	8.851
10.08	66.75	4.333	7.431
10.08	64.25	3.333	5.129
10.08	62.00	3.000	4.169
10.08	59.75	3.563	3.162
$10^2[K_2\ S_2\ O_8] = 10^2\ [KI] = 3.08\ (mol./dm^3)$			
12.32	69.00	6.066	9.332
12.32	66.75	5.867	8.318
12.32	64.25	3.533	4.467
12.32	62.00	3.492	3.935
12.32	59.75	3.432	3.388

$10^2[K_2\ S_2\ O_8] = 10^2\ [KI] = 5.04\ (mol./dm^3)$

20.16	69.00	8.108	7.499
20.16	66.75	6.900	5.623
20.16	64.25	5.283	3.715
20.16	62.00	4.398	2.692
20.16	59.75	4.385	2.399

$10^2[K_2\ S_2\ O_8] = 10^2\ [KI] = 7.00\ (mol./dm^3)$

28.00	69.00	8.903	5.370
28.00	66.75	7.433	3.890
28.00	64.25	6.583	2.884
28.00	62.00	5.000	1.738
28.00	59.75	4.842	1.493

The effect of dielectric constant on rate constant can be explained by the Laidler-Eyring equation [6] for single and double sphere models respectively. The equations are expressed as followed.

$$\ln k_0 = \ln k_\infty - \frac{e^2}{2\varepsilon KT}\left[\frac{Z^2{}_A}{r^{\neq}} - \frac{Z^2{}_A}{r_A} - \frac{Z^2{}_B}{r_B}\right] \quad \text{(for single sphere mode) ...(6)}$$

$$\ln k_0 = \ln k_\infty - \frac{Z_A.Z_B.e^2}{KTr_{AB}}.\frac{1}{\varepsilon} \quad \text{(for double sphere model)...(7)}$$

where k_∞ is the rate constant at zero ionic strength and infinite dielectric constant. $r^{\neq}$ and r_{AB} are the radii of the activated complex for the single model and the double sphere model, respectively, and r_A, r_B are radii of the ions A and B, respectively. The plots of logarithm of rate constant at zero ionic strength ($\log k_o$) against the reciprocal of the dielectric constant of the medium ($1/\varepsilon$) at various ionic strengths and temperatures were straight lines with negative slopes. A representative graph is shown in Figure - 2.

The experimental values of $r^{\neq}$ and r_{AB} calculated from the slopes of straight lines using equations (6) and (7) are tabulated in Table - 3.

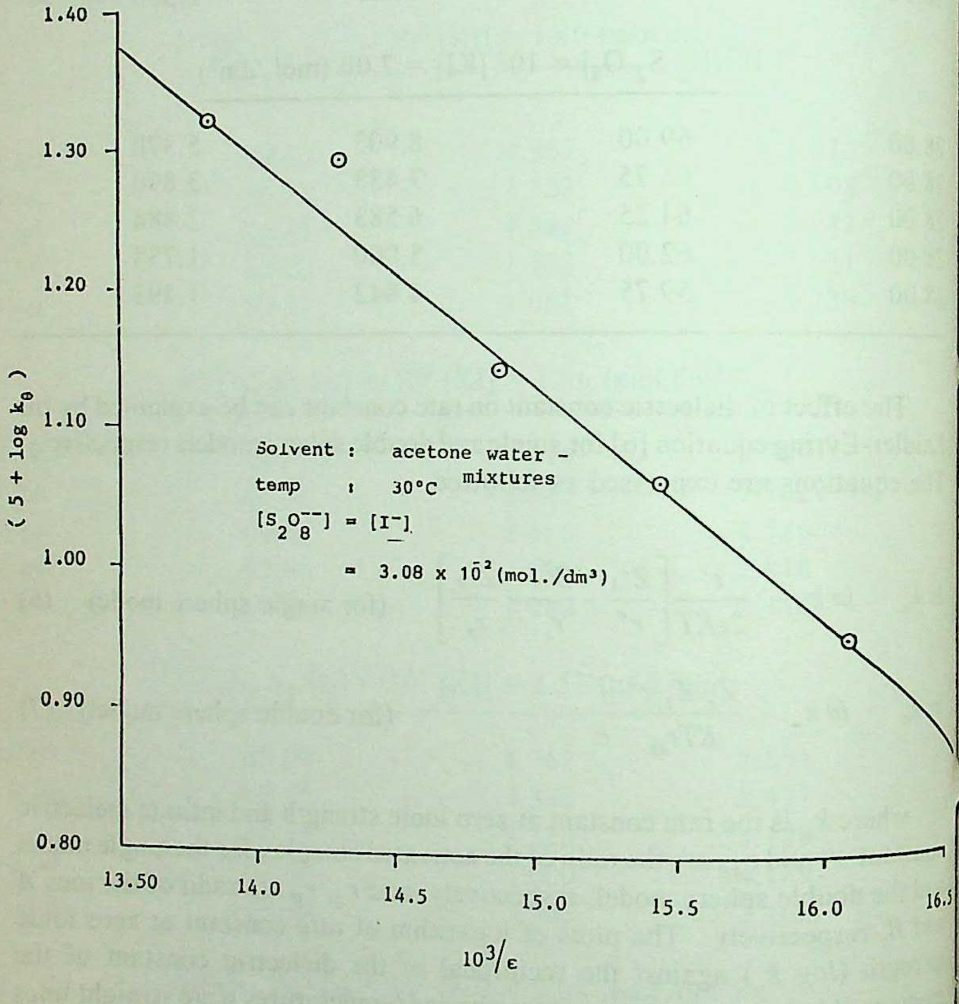

FIGURE - 2

Fig. 2 : Plot of log k_o versus '$1/\varepsilon$ 'in aqueous acetone medium at 30°C.

TABLE - 3 : CALCULATED VALUES OF RADII OF ACTIVATED COMPLEX

$10^2[K_2S_2O_8]$ (mol/dm^3)	10^2[KI] (mol/dm^3)	$r^{\neq}$ (Å)	r_{AB} (Å)
	Temp : 30°C		
1.40	1.40	3.04	3.74
1.96	1.96	2.23	1.86
2.52	2.52	2.25	1.89
3.08	3.08	2.70	2.77
5.05	5.04	2.26	1.91
7.00	7.00	2.19	1.81
		Average	Average
		2.44 ± 0.37	2.33 ± 0.83
	Temp : 40°C		
1.40	1.40	2.57	2.48
1.96	1.96	2.13	1.72
2.52	2.52	2.49	2.33
3.08	3.08	2.44	2.22
5.05	5.04	2.30	1.98
7.00	7.00	2.20	1.83
		Average	Average
		2.35 ± 0.2	2.09 ± 0.32

The radius of the peroxodisulphate ion was estimated on the assumption that the volume of the molecule of a compound which is supposed to be spherical is a sum of volumes of individual ions consituting the compound as follows :

$$V_{S_2O_8^{2-}} = 2V_{S}6 + + 8V_{O^{2-}}$$

where V is the volume of the ion.

$$\frac{4}{3}\pi r^3_{S_2O_8^{2-}} = 2(\frac{4}{3}\pi r_S^3 6 + \; + 8(\frac{4}{3}\pi r^3_{O^{2-}})$$

$$r_{S_2O_8^{2-}} = 2.80 \text{ Å}$$

$$r_{I^-} = 2.16 \text{ Å (taken from Emeleus [19]} \qquad \text{........(11)}$$

The radius of activated complex for single sphere model ($r^{\neq}$) is calculated on the basis of assumption that the volume of single sphere activated complex ($V^{\neq}$) is the sum of volume of reactants.

$$V^{\neq} = V_{I^-} + V_{S_2O_8^{2-}} \qquad \text{........(12)}$$

$$r^{\neq} = 3.176 \text{ Å} \qquad \text{........(13)}$$

The radius of activated complex for double sphere model is calculated assuming that the radius of activated complex is the sum of the radii of reactants.

$$r_{AB} = r_{S_2O_8^{2-}} + r_{I^-} \qquad \text{........(14)}$$

$$= 2.80 + 2.16 = 4.96 \text{ Å} \qquad \text{........(15)}$$

In all calculations the values of radii of individual ions were taken from Emeleus [19] and confirmed by Wilson [20]. It has been concluded from the comparison of average experimental values of $r^{\#}$ and r_{AB} with their theoretical values that the shape of the activated complex could best be given by single sphere model for reaction between peroxodisulphate and iodide ions because the value of $r^{\#}_{expt}$ is more closer to the theoretical value of single sphere model. The results show that change in temperature has no effect on the shape of activated complex. The values of $r^{\#}$ and r_{AB} are slightly less than the values given by Ghaziuddin et al. [14]. This may be due to the volatile nature of acetone.

Electrostatic contributions to the changes in free energy of activation $\Delta G^{\#}_{es}$ in the formation of activated complex for single sphere model were evaluated using the following expression [5] :

$$\Delta G^{\neq}_{e.s} = \frac{N_A e^2}{2}\left[\frac{(Z_A + Z_B)^2}{r^{\neq}} - \frac{Z^2{}_A}{r_A} - \frac{Z^2{}_B}{r_B}\right] \qquad \ldots\ldots(16)$$

The values of $\Delta G^{\neq}_{es}$ at different temperatures i.e. 30^0 and $40\ ^0$C and ionic strengths as a function of dielectric constant are summarised in Table - 4.

ABLE - 4 : ELECTROSTATIC CONTRIBUTION TO $\Delta G_{es}^{\neq}$

$10^2\ \mu$ (mol/dm^3)	Dielectric constant (ε)				
	72.25	69.75	67.25	64.75	62.20
			Temp : 30°C		
5.60	1.028	1.065	1.105	1.147	1.194
7.84	2.062	2.136	2.215	2.300	2.394
10.08	2.027	2.100	2.178	2.262	2.235
12.32	1.386	1.436	1.489	1.547	1.610
20.16	2.010	2.083	2.160	2.243	2.335
28.00	2.133	2.209	2.291	2.380	2.477
			Dielectric constant (ε)		
	69.00	66.75	64.25	62.00	59.75

Temp : 40°C $_{e.s}$(KJ/mole)

5.60	1.621	1.676	1.741	1.804	1.872
7.84	2.349	2.429	2.523	2.615	2.713
10.08	1.735	1.793	1.863	1.930	2.003
12.32	1.809	1.870	1.943	2.014	2.089
20.16	2.035	2.103	2.186	2.265	2.350
28.00	2.214	2.289	2.378	2.464	2.555

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp. 119-132

ON A LAPLACE - HARDY TRANSFORMATION

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(Received 04.09.1992)

ABSTRACT

In this paper we shell give a few funuamental theorems that depict certain properties of Laplace-Hardy transformation that seem to be interesting and significant.

INTRODUCTION

In the present paper for the Classical Laplace-Hardy transformation, we shall use the notation:

$$(1).\ \phi[f: p.q] = pq\int_0^\infty\int_0^\infty e^{-pxqy} C_\upsilon(pxqy) f(px, qy)\,dxdy$$

where $f(px, qy)$ is suitably chosen function under a certain set of condition on ν and α given by Pathak and Pandey, [5].
and

$$(2).\ C_\nu(z) = \cos(a\pi) J_\nu(z) + \sin(a\pi) Y_\nu(z)$$

$J_\nu(z)$ and $Y_\nu(Z)$ being Bessel functions [(2), p.38] of the first and second kind respectively, and (2) valid under certain conditions given by Erdelyi [2].

Throughout this work we use the notation:

$$(3).\ \phi[f : p,q] \underset{=}{C_\nu} f(px, qy).$$

The C_ν-function is defined by (2), and if we take $a = 0$, we obtain $C_\nu(z) = J_\nu(z)$ And If $a = 1/2$, we obtain $C_\nu(z) = y_\nu(z)$. And if $\nu = 1/2$, we obtain $C_\nu(z$ $= [(1/2)\pi z]^{-1/2} \sin(z - a\pi)$. So by using the conditions on 'ν' and 'a' we obtain some special cases of Hardy's transform.

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Theorem

If $\phi_i\,[f_i : p.q\,] \overset{C_\nu}{\underset{\cdot\cdot}{=}} f_i\,(px.\ qy),\ i = 1.2.$

then

$$(4).\quad \int_0^\infty\!\!\int_0^\infty \phi_1[f_1 : p.q]\,f_2(px.qy)\,\frac{dp}{p}\,\frac{dp}{q}$$

$$= \int_0^\infty\!\!\int_0^\infty f_1(px.qy)\ \phi_2[f_2 : p.q]\,\frac{dp}{p}\,\frac{dp}{q}$$

provided that the integral involved are absolutely convergent.

Proof

By putting the value of $\phi_1\,[\,f_1 : p,q\,]$ on the left hand side of (4), we have

$$\int_0^\infty\!\!\int_0^\infty \{\, pq\int_0^\infty\!\!\int_0^\infty e^{-pxqy}C_\nu(pxqy)\,f_1(px.qy)\,dxdy\,\}\,f_2(px.qy)\,\frac{dp}{p}\,\frac{dp}{q}$$

Now changing the order of integration, we get

$$= \int_0^\infty\!\!\int_0^\infty \{\, pq\int_0^\infty\!\!\int_0^\infty e^{-pxqy}C_\nu(px.qy)\,f_2(px.qy)\,\frac{dp}{p}\,\frac{dp}{q}\,\}\ f_1(px.qy)\,dxdy.$$

$$= \int_0^\infty\!\!\int_0^\infty f_1(px.qy)\{\int_0^\infty\!\!\int_0^\infty e^{-pxqy}C_\nu(pxqy)\,f_2(px.qy)\,dp\,dq\,\}\,dx\,dy$$

$$= \int_0^\infty\!\!\int_0^\infty f_1(px.qy)\{\,xy\int_0^\infty\!\!\int_0^\infty e^{-pxqy}C_\nu(pxqy)\,f_2(px.qy)\,dp\,dq\,\}\,\frac{dx}{x}\,\frac{dy}{y}$$

$$= \int_0^\infty\!\!\int_0^\infty f_1(px.qy)\ \phi_2[f_2 : p.q]\,\frac{dx}{x}\,\frac{dy}{y}$$

$$= \int_0^\infty\!\!\int_0^\infty f_1(px.qy)\ \phi_2[f_2 : p.q]\,\frac{dp}{p}\,\frac{dq}{q}$$

Hence the theorem.

Theorem 2

Let $\phi\,[f : p.q] \overset{C_\nu}{\underset{\cdot\cdot}{=}} f(px.qy) = g(p.q)$

and $\phi[h:p,q] \underset{\cdot\cdot}{\overset{C_\upsilon}{=}} h(px.qy) = \psi(p.q)$

then

$$(5).\quad \int_0^\infty \int_0^\infty \psi(u,v)\, f(ux,vy)\, (uv)^{-1} du\, dv$$

$$= \int_0^\infty \int_0^\infty g(u,v)\, h(ux,vy)\, (uv)^{-1} du\, dv$$

provided that either of the integral in (5) is absolutely convergent and the ϕ-transformation of $|f(ux, vy)|$ and $|h(ux.vy)|$ exist.

Proof

In view of theorem 1:

$$(6).\ \psi(u.v) = uv \int_0^\infty \int_0^\infty e^{-uxvy} C_\upsilon(uxvy)\, h(ux.vy)\, dx\, dy$$

Now putting the value $\psi(u.v)$ on the left hand side of (5) :

$$\int_0^\infty \int_0^\infty \{uv \int_0^\infty \int_0^\infty e^{-uxvy} C_\upsilon(uxvy)\, h(ux.vy)\, dx\, dy\} f(ux.vy)(uv)^{-1} du\, dv$$

changing the order of integration:

$$= \int_0^\infty \int_0^\infty \{uv \int_0^\infty \int_0^\infty e^{-uxvy} C_\upsilon(uxvy)\, f(ux.vy)\, (uv)^{-1} du\, dv\}\, h(ux.vy) dx\, dy$$

$$= \int_0^\infty \int_0^\infty \{xy \int_0^\infty \int_0^\infty e^{-uxvy} C_\upsilon(uxvy)\, f(ux.vy)\, du\, dv\}\, h(ux.vy)\, (xy)^{-1} dx\, dy$$

$$= \int_0^\infty \int_0^\infty g(x,y)\, h(ux,vy)\, (xy)^{-1}\, dx\, dy$$

$$= \int_0^\infty \int_0^\infty g(u,v)\, h(ux,vy)\, (uv)^{-1}\, du\, dv.$$

Hence the theorem.

Presently numerical inversion of certain integral transforms are being studied in order to make them computer oriented. [6].

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Journal of Natural & Physical Sciences Vol. 5(1991-94) pp. 123-137

ON A NONLINEAR FUNCTIONAL INTEGRODIFFERENTIAL SYSTEM

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(Received 08.09.1992)

ABSTRACT

The aim of the present paper is to study the existence, uniqueness and other properties of the solutions of a nonlinear functional integrodifferential system of more general type by using monotone iterative method, comparison principle, the notion of upper and lower solutions and integral inequality established by Pachapatte.

Mathematical Subject Classification: 47HO5 47H10

Keywords and Phrases: Integrodifferential System, Monotone iterative technique.

INTRODUCTION

In what follows, we denote by R^n the *n-dimensional Euclidean space* with the Euclidean norm $\|\cdot\|$ .Let $I_o = [-r, 0]$, $I = [t_0, t_o+T]$, $J = [t_0 - r, t_0+T]$, $r > 0$, $t_o \geq 0$ and $T > o$ ar econstants. Let $C = C[I_0, R^n]$ be the Banach space of continuous functions defined on I_o with supremum norm $\|\cdot\|_c$. We use the notation $< v, w >$ to denote the segment in $C[J, R^n]$ defined by $<v, w> = \{x \in C[J, R^n] : v(t) \leq x(t) \leq w(t),\ t \in J\}$. If x is a continous function from J to R^n and $t \in I$, then x_t denotes the element of C given by $x_t(\theta) = x(t+\theta)$ for $\theta \varepsilon I_0$ In the present paper, we investigate the existence, uniqueness and other properties of the solution of following nonlinear functional intergro differential system

$$x'(t) = f\left(t, x(t), x_t, \int_{t_0}^{t} k(t, s, x(s), x_s)\,ds\right), \quad t \in I, \tag{1.1}$$

$$x_{t_0}(\theta) = \phi(\theta), \quad \theta \in I_0$$

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where $x : J \to R^n$, $K \in C[I \times I \times R^n \times C, R^n]$, $f \in C[I \times R^n \times C \times R^n, R^n]$ and $0 \in C[I_0, R^n]$ is a given function.

The problems of existence, uniqueness and other properties of solutions of special forms of the system (1.1) have been studied by many authors in the literature by using different techniques, for example, see [2,4,6,7,9,10,12] and the references given therein. In [9], V. Lakshmikantham and B.G. Zhang have proved the existence of extremal solution of the equation (1.1) when k = 0 by using monotone iterative technique. B.G. Pachpatte [12] has established the existence of extermal solutions of the system (1.1) without functional argument. The main tools employed in our analysis are based on the monotone iterative method, comparision principle, the notion of upper and lower solutions and integral inequality established by Pachpatte.

We organize the paper as follows: In section 2, we present the preliminaries and statement of our main results, the proofs of which are contained in sections 3 and 4.

PRELIMINARIES AND STATEMENT OF RESULTS

Before proceeding to the statement of our main results, we shall setforth some preliminaries and hypotheses on the functions involved in (1.1) that will be used in our subsequent discussion. Throughout this paper, without further mention, we assume that all inequalities between vectors are componentwise

Definition 1. The function $w \in C[J, R^n] \cap C'[I, R^n]$ is called an *upper solution* of (1.1.) if

$$w'(t) \geq f\left(t, w(t), w_t, \int_{t_0}^{t} k(t, s, w(s), w_s)\,ds\right), \quad t \in I$$

$$w_{t_0}(\theta) \geq \phi(\theta), \quad \theta \in I_0$$

Similarly, a function $v \in C[J, R^n] \cap C'[I, R^n]$ is called a *lower solution of* (1.1) if

$$v'(t) \leq f\left(t, v(t), v_t, \int_{t_0}^{t} k(t, s, v(s), v_s)\,ds\right), \quad t \in I,$$

$$v_{t_0}^{\cdot}(p) \le \phi(\theta). \quad \theta \in I_0$$

Definition 2. The functions α and β, $\alpha, \beta \in C[J,R^n] \cap C'[I,R^n]$ are called minimal and maximal solutions of (1.1) respectively, if every other solution $x \in C[J,R^n] \cap C'[I,R^n]$ of (1.1) satisfies the relation

$$\alpha(t) \le x(t) \le \beta(t), \ t \varepsilon J.$$

Definition 3. The solution $x(t)$ of (1.1) is said to be uniformly slowly growing if for every $\gamma > 0$ there exists a constant Q, possibly depending on γ, such that the indequality

$$\|x_t\|_C \le Q \, \|\phi\|_C \cdot e^{\gamma(t-t_0)} \, , t \ge t_0$$

holds for $\|\phi\|_C < \infty$

we require the following lemmas in our further discussion.

Lemma 1. (See, [9,p.228]). Assume that $u \in C[J,R] \cap C''[I,R]$ and

$$u'(t) \le -Mu(t) - N \int_{-r}^{0} u_t(s)ds,$$

$t \in I$. Further assume that

i) either $u(t_o) \le u_{t_o}(s) \le 0$, $s \in I_0$ and $(M + Nr)\, T \le 1$
or

ii) $u_{t_o}(s) \le 0$, $s \in I_0$, $u \in C'([t_0 - r, t_0], R)$ and

$$u'(t) \le \frac{\lambda}{T+r}, \ t \varepsilon [t_0 - r, t_0]$$

where $\min_{[t_0-r;t_0]} u(t) \quad -\lambda$, $\lambda \ge 0$ and $(M+Nr)(T+r) \le 1$. Then $u(t) \le 0$ on I.

Lemma 2. (See [11. p.758)]. *Let $u(t)$, $m(t)$ and $n(t)$ be real* valued nonnegative continuous functions defined on R^+, for which the inequality

$$u(t) \leq u_0 + \int_0^t m(s)\,u(s)\,ds + \int_0^t m(s)\left[\int_0^s n(\tau)\,u(\tau)\,d\tau\right]ds.$$

holds for all $t \in R^+$, where u_0 is a nonnegative constant. Then

$$u(t) \leq u_0\left[1 + \int_0^t m(s)\exp\left\{\int_0^s \big(m(\tau) + n(\tau)\big)\,d\tau\right\}ds\right].$$

for all $t \in R^+$

For convenience, we list the following hypotheses.

(H_1) The functions $v, w \in C. [J, R^n] \cap C' [I, R^n]$ with $v(t) \leq w(t)$ on J are lower and upper solutions of (1.1).

(H_2) For each $i, 1 \leq i \leq n$, $k_i\,(t,s,x,\phi)$ is *monotone nondecreasing* in x and ϕ for fixed t and s, and $f_i(t,x,\phi,u)$ is *monotone nondecreasing* in u for fixed t, x and ϕ.

(H_3) For each $i, 1 \leq i \leq n$, there exist constants $M, N, \geq 0$ such that

$$f_i\,(t, x, \phi, u) - f_i\,(t, \bar{x}, \bar{\phi} u)$$

$$\geq -M\left(x_i - \bar{x}_i\right) - N\int^{0}\left[\phi_i(s) - \bar{\phi}_i(s)\right]ds$$

whenever $v(t) \leq \bar{x} \leq x \leq w(t)$ and $v_t \leq \bar{\phi} \leq \Phi \leq w_t$ for $t \in I$.

(H_4) The difference $v_{t_0} - \phi$, $\phi - w_{t_0}$ satisfy either the assumption (i) or (ii) of lemma 1

(H_5) There exist nonegative constants M_1, M_2, M_3, M_4 and M_5 such that

(2.1) $\| K(t,s,x,\phi) - k(t,s,\bar{x}\,\bar{\phi}) \| \le M_1 // x - \bar{x} \| + M_2 \| \phi - \bar{\phi} \|_c$

(2.2) $\| f(t,x,\phi,u) - f(t,\bar{x}\,\bar{\phi},\ \bar{u}) \|$
$$\le M_3 \| x - \bar{x} \| + M_4 \| \phi - \bar{\phi} \|_C + M_4 \| u - \bar{u} \|$$

(H_6) *For* $t,\ s \in [t_0, \infty),\ \phi \in C,\ x,\ u \in R^n$,

(2.3) $\| K(t,s,x,\phi) \| \le L_i(s)\ [\ \| x \| + \| \phi \|_C\ ]$

(2.4) $\| f(t,x,\phi,u) \| \le L_2(t)\ [\ \| x \| + \| \phi \|_C + \| u \|\ ]$

where $L_1,\ L_2 \in C([\,t_0\ \infty),\ [\,t_0\ \infty))$
and

(2.5) $\int_{t_0}^{\infty} L_1(s)ds < \infty,\ \int_{t_0}^{\infty} L_2(s)ds < \infty.$

(H_7) For $t,\ s \in [t_0, \infty],\ \gamma > 0,\ \phi \in C,\ x,\ u \in R^n$,

(2.6) $\| K(t,s,x,\phi) \| \le e^{-\gamma(s-t)} L_1(s)\ [\ \| x \| + \| \phi \|_C\ ]$

(2.7) $\| f(t,x,\phi,u,) \| \le e^{-\gamma t} L_2(t)\ [\ \| x \| + \| \phi \|_C + \| u \|\ ]$

where $L_1,\ L_2 \in C([t_0, \infty\ ),\ [t_0, \infty\ ))$ and (2.5) holds.

Our main results are established in the following theorems.

THEOREM 1. Let the hypotheses (H_1) - (H_4) hold. Then there exists monotone nondecreasing sequence $\{v_n\}$ and monotone non-increasing sequence $\{w_n\}$ such that $v_n \to \alpha,\ w_n \to \beta$ as $n \to \infty$ uniformly on I and that α,β are minimal and maximal solutions of the system (1.1) respectively. That is , if x is any solution of (1.1)in $<v,w>$, then

$$v \le v_1 \le \ldots \le v_n \le \ldots \le \alpha \le x \le \beta \le \ldots \le w_n \le \ldots \le w_1 \le w$$

THEOREM 2. Let the hypotheses (H_1) - (H_5) hold. Then the maximal solution $\beta(t)$ and the minimal solution $\alpha(t)$ obtained in Theorem 1 coincide on J, that is $\beta(t) = \alpha(t)$, *for* $t \in J$.

In the following theorems, we assume that solutions of the system (1.1) exist on $[t_o - r, \infty)$.

THEOREM 3. Let the hypothesis (H_6) holds. Then any solution $x(t)$ of the system (1.1) is bounded on $[t_o - r, \infty)$.

THEOREM 4 Let the hypothesis (H_7) holds. Then any solution $x(t)$ of the system (1.1) is uniformly slowly growing.

Remark 1. We note that in [8], G.S. Ladde and B.G. Pachpatte, have established existence of the solution of the equation of type (1.1) by using different conditions and techniques. It is also important to note that M.B. Dhakne and B.G. Pachpatte in [5] have investigated the problems of boundedness, stability, asympotic behaviour and other properties of the solutions of an abstract functional integro-differential equation by using the semigroups method.

PROOFS OF THEOREMS 1 AND 2

For any $\eta \,\varepsilon\, C\,[J, R^n]$ such that $v \leq \eta \leq w$ on J, we consider the following linear system

$$x'(t) = f\left(t, \eta(t), \eta_t, \int_{t_0}^{t} k(t, s, \eta(s), \eta_s)\,ds\right) - M\,[x(t) - \eta(t)]$$

$$(3.1) \qquad -N \int_{-r}^{0} \left[x_t(s) - \eta_t(s)\right] ds, \quad t \in I,$$

$$x_{t_0}(\theta) = \phi(\theta), \; \theta \in I_0$$

Clearly, the linear system (3.1) possesses a unique solution $x(t)$ on J, for each η. We, now, define a mapping A by $A\eta = x$.

We prepare a lemma which forms the basis of our main result.

Lemma 3. Let the hypotheses (H_1) - (H_4) hold. Then

(i) $v \leq Av$, $w \geq Aw$ on J;

(ii) A is a monotone operator on the segment $<v, w>$

Proof : Set $Aw = x$ where x is the unique solution of (3.1) corresponding to w, and $z_i(t) = x_i(t) - w_i(t)$ for each i, $1 < i < n$. Then, we have

$$z'_i(t) = x'_i(t) - w'_i(t)$$

$$\leq f_i\left(t, w(t), w_t, \int_{t_0}^{t} k(t,s,w(s),w_s)ds\right) - M[x_i(t) - w_i(t)]$$

$$-N\int_{-r}^{0}\left[x_{i_t}(s) - w_{i_t}(s)\right]ds - f_i\left(t, w(t), w_t, \int_{t_0}^{t} k(t,s,w(s),w_s)ds\right)$$

$$= -Mz_i(t) - N\int_{-r}^{0} z_{i_t}(s)\,ds$$

and $z_{i_{t0}} = \phi_i - w_{i_{t0}} \leq 0$. By an application of lemma1, $z_i(t) = x_i(t) - w_i(t) \leq 0$ on J, that is, for each i, $1 \leq i \leq n$, $w_i(t) \geq x_i(t)$ for $t \in J$. Hence, $w(t) \geq x(t) = Aw(t)$ on J. Similarly, we show that $v \leq Av$ on J. This proves(i).

Let $p, q \in <v, w>$ and $p \leq q$. Suppose that $Ap = x$ and $Aq = y$. For each $i, 1 \leq i \leq n$, set $z_i(t) = x_i(t) - y_i(t)$. Then

$$z_i'(t) = x_i'(t) - y'_i(t)$$

$$= f_i(t, p(t), p_t, \int_{t_0}^{t} k(t, s, p(s), p_s)\, ds) - M[x_i(t) - p_i(t)]$$

$$-N\int_{-r}^{0}[x_{i_t}(s) - p_{i_t}(s)]\,ds - f_i(t, q(t), q_t, \int_{t_0}^{t} k(t, s, q(s), q_s)\, ds)$$

$$+ M[y_i(t) - q_i(t)] + N\int_{-r}^{0}\left[y_{i_t}(s) - q_{i_t}(s)\right]ds$$

$$\leq f_i(t, p(t), p_t, \int_{t_0}^{t} k(t, s, q(s), q_s)\, ds.$$

$$- f_i(t, q(t), q_t, \int_{t_0}^{t} k(t, s, q(s), q_s)\, ds) - M[x_i(t) - p_i(t)]$$

$$- N \int_{-r}^{0} [\, x_{i_t}(s) - p_{i_t}(s)]\, ds \;+ M\, [y_i(t) - q_i(t)]$$

$$+ N \int_{-r}^{0} [\, y_{i_t}(s) - q_{i_t}(s)]\, ds$$

$$= - Mz_i(t) - N \int_{-r}^{0} z_{i_t}(s)\, ds$$

and $z_{i_{t_0}} = \phi_i - \phi_i = 0$. By lemma 1, $z_i(t) = x_i(t) - y_i(t) \leq 0$ on I, that is for each i, $1 \leq i \leq n$, $x_i(t) \leq y_i(t)$ for $t \in J$. Hence, $x(t) \leq y(t)$ on J. Thus, A is a monotone operator on $<v,w>$. This completes the proof of lemma 3.

In order to prove Theorem 1, define the sequences $\{ v_n \}$ and $\{ w_n \}$ as follows :

$$v_1 = Av,\ v_n = Av_{n-1}\,,\ n = 2,3,\ldots$$
$$w_1 = Aw,\ w_n = Aw_{n-1}\,,\ n = 2,3,\ldots$$

on J. by an application of lemma 3, we get

$$v \leq v_1 \leq v_2 \leq \ldots \leq v_n \leq w_n \ldots \leq w_2 \leq w_1 \leq w$$

on J. Using standard arguments (see [1], [3]), it is easy to show that

$$\lim_{n\to\infty} v_n = \alpha \ \text{ and } \lim_{n\to\infty} w_n = \beta$$

uniformly on J and that α, β are solutions of the system (1.1) on J.

Let $x(t)$ be any solution of the system (1.1) such that $v(t) \leq x(t) \leq w(t)$ on J. For each i, $1 \leq i \leq n$, set $z_i(t) = v_{1_i}(t) - x_i(t)$. Then

$$z'_i(t) = v_{1_i}{}'(t) - x'_i(t)$$

$$= f_i(t,\ v(t),\ v_t,\ \int_{t_0}^{t} k(t,\ s,\ v(s),\ v_s)\, ds) - M\, [v_{1_i}(t) - v_i(t)]$$

$$-N\int_{-r}^{0} [v_{1_{i_t}}(s) - v_{i_t}(s)]\, ds - f_i\left(t, x(t), x_t, \int_{t_0}^{t} k(t, s, x(s), x_s)\, ds\right)$$

$$\le f_i\left(t, v(t), v_t, \int_{t_0}^{t} k(t, s, x(s), x_s)\, ds\right)$$

$$- f_i\left(t, x(t), x_t, \int_{t_0}^{t} k(t, s, x(s), x_s)\, ds\right)$$

$$- M[v_{1_i}(t) - v_i(t)] - N\int_{-r}^{0} [v_{1_{i_t}}(s) - v_{i_t}(s)]\, ds$$

$$\le M[x_i(t) - v_i(t)] + N\int_{-r}^{0} [x_{i_t}(s) - v_{i_t}(s)]\, ds$$

$$- M[v_{1_i}(t) - v_i(t)] - N\int_{-r}^{0} [v_{1_{i_t}}(s) - v_{1_i}(s)]\, ds$$

$$= - Mz_i(t) - N\int_{-r}^{0} z_{i_t}(s)\, ds$$

and $z_{i_{to}} = v_{1_{to}} - x_{i_{to}} = \phi_i - \phi_i = 0$ Then by an application of Lemma 1, we obtain $z_i(t)$ v_{1_i} 0 on J.That is : for each i, $1\le i \le n$, $v_{1_i}(t) \le x_i(t)$, $t \in J$. Hence $v_1(t) \le x(t)$ for $t \in J$. By induction, it can be shown that $v_n \le x$ for every $n \ge 2, 3, \ldots$ .Similarly, $x \le w_n, n = 1, 2, \ldots$ .Therefore, $v_n \le x \le w_n$ and hence $\alpha \le x(t) \le \beta$ on J.That is : α, β are minimal and maximum solutions of the system (1.1) respectively. This completes the proof of the Theorem 1.

The function $\beta(t)$ and $\alpha(t)$ are solutions of the system (1.1). so we have,

$$\beta(t) = \phi(0) + \int_{t_0}^{t} f\left(s, \beta(s), \beta_s, \int_{t_0}^{s} k(s, \tau, \beta(\tau), \beta_\tau)\, d\tau\right) ds,\ t \in I, \tag{3.2}$$

$$\beta(t) = \phi(t-t_0),\ t_0 - r \le t \le t_0$$

and

$$\alpha(t) = \phi(0) + \int_{t_0}^{t} f(s,\ \alpha(s),\ \alpha_s,\ \int_{t_0}^{s} k(s,\ \tau,\ \alpha(\tau),\ \alpha_\tau)\, d\tau) ds,\ t \in I,$$

(3.3)

$$\alpha(t) = \phi(t-t_0).\ t_0 - r \le t \le t_0$$

Define $z(t) = \|\beta(t) - \alpha(t)\|$, $t \in J$, Clearly, $z(t) = 0$ for $t_0 - r \le t \le t_0$. Now for $t \in I$, we obtain from (3.2), (3.3) and using (2.1) and (2.2),

$$\begin{aligned}
z(t) &= \| \beta(t) - \alpha(t)\| \\
&\le \int_{t_0}^{t} \| f(s, \beta(s), \beta_s, \int_{t_0}^{s} k(s, \tau, \beta(\tau)\, \beta_\tau)\, d\tau) \\
&\quad - f(s, \alpha(s), \alpha_s, \int_{t_0}^{s} k(s, \tau, \alpha(\tau)\, \alpha_\tau) d\tau \| \, ds \\
&\le M_3 \int_{t_0}^{t} \|\beta(s) - \alpha(s)\|_C\, ds + M_4 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C\, ds \\
&\quad + M_1 M_5 \int_{t_0}^{t}\int_{t_0}^{s} \| \beta(\tau) - \alpha(\tau) \|_C\, d\tau\, ds + M_2 M_5 \int_{t_0}^{t}\int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C\, d\tau\, ds \\
&\le M_3 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C\, ds + M_4 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C\, ds \\
&\quad + M_1 M_5 \int_{t_0}^{t}\int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C\, d\tau\, ds + M_2 M_5 \int_{t_0}^{t}\int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C\, d\tau\, ds \\
&= M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C\, ds + M_6 \int_{t_0}^{t}\int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C\, d\tau\, ds
\end{aligned}$$

where $M_6 = \max\ \{M_3 + M_4,\ (M_1 + M_2)\, M_5\}$. Thus $t \in I$, we have

$$\text{(3.4)} \quad \| \beta(t) - \alpha(t)\| \le M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C\, ds + M_6 \int_{t_0}^{t}\int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C\, d\tau\, ds$$

Case 1.

Suppose $t \ge t_0 + r$. Then for all $\theta \in [-r, 0]$, $t + \theta \ge t_0$. For such θ's from (3.4), we get

$$\|\beta(t+\theta) - \alpha(t+\theta)\| \leq M_6 \int_{t_0}^{t+\theta} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t+\theta} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C \, d\tau \, ds$$

$$\leq M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \|\beta_\tau - \alpha_\tau\|_C \, d\tau \, ds$$

which yields

$$\text{(3.5)} \qquad sup_{\theta \in [-r,\, 0]} \|\beta(t+\theta) - \alpha(t+\theta)\| = \| \beta_t - \alpha_t \|_C$$

$$\leq M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C \, d\tau \, ds$$

Case **2.**

Suppose $t_0 \leq t < t_0 + r$. Then for all $\theta \in [-r, t_0, -t]$, we have $t + \theta < t_0$. we observe that

$$\text{(3.6)} \quad \| \beta(t + \theta) - \alpha(t + \theta) \| = 0$$

For $\theta \in [t_0 - t,\ 0], t + \theta \geq t_0$. Then, we get as in the case 1,

$$\text{(3.7)} \quad \| \beta(t + \theta) - \alpha(t + \theta) \|$$

$$\leq M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C \, d\tau \, ds$$

Thus, for every $\theta \in [-r,\ 0]$, $(t_0 \leq t < t_0 + r)$, from (3.6) and (3.7), have

$$\| \beta(t + \theta) - \alpha(t + \theta) \| \leq M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C \, d\tau \, ds$$

which yields

$$\text{(3.8)} \quad \| \beta_t - \alpha_t \|_C \leq M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C \, d\tau \, ds$$

For every $t \in I$, from (3.5) and (3.8), we have

$$(3.9)\quad \| \beta_t - \alpha_t \|_C \leq M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C d\tau \, ds$$

$$\leq \varepsilon + M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C d\tau \, ds$$

where $\varepsilon > 0$ is an arbitrary constant. Now, by applying Lemma 2 with $u(t) = \| \beta_t - \alpha_t \|_C$ we get from (3.9)

$$\| \beta_t - \alpha_t \|_C \leq \varepsilon \, [1 + M_6 \exp\{(M_6 + 1)T\} T]$$

Since ε is arbitrary, $\| \beta_t - \alpha_t \|_C = 0$, $t \, \varepsilon \, I$, which implies that $\| \beta(t) - \alpha(t) \| = 0$, $t \in I$ i.e. $\beta(t) = \alpha(t)$, $t \in I$. Hence,

$$\beta(t) = \alpha(t), \; t \in J$$

This complete the proof of the Theorem 2.

PROOFS OF THEOREMS 3 AND 4

The solution of the system (1.1) on $(t_0 - r, \infty)$ is given by

$$x(t) = \phi(0) + \int_{t_0}^{t} f(s, x(s), x_s, \int_{t_0}^{s} k(s, \tau, x(\tau), x_\tau) \, d\tau) \, ds \qquad \text{for } t \geq t_0$$

$$(4.1)$$

$$= \phi(t - t_0), \qquad \text{for } t_0 - r \leq t \leq t_0$$

If $t \geq t_0$, then from (4.1) and using (2.3), (2.4), we get

$$(4.2)\quad \|x(t)\| \leq \|\phi\|_C + \int_{t_0}^{t} L_2(s) [\| x \| + \| x_s \|_C] \, ds$$

$$+ \int_{t_0}^{t} L_2(s) \int_{t_0}^{s} L_1(\tau) \, [\| x \| + \| x_\tau \|_C] \, d\tau \, ds$$

$$\leq \|\phi\|_c + \int_{t_0}^{t} 2L_2(s) \, \| x_s \|_c \, ds + \int_{t_0}^{t} 2L_2(s) \int_{t_0}^{s} L_1(\tau) \, \| x_\tau \|_C \, d\tau \, ds$$

By proceeding as in the proof of Theorem 2, we get from (4.2)

$$(4.3)\ \|x_t\|_C \le \|\phi\|_C + \int_{t_0}^{t} 2L_2(s)\,\|x_s\|_C\,ds + \int_{t_0}^{t} 2L_2(s)\int_{t_0}^{s} L_1(\tau)\,\|x_\tau\|_C\,d\tau\,ds$$

Applying Lemma 2 with $u(t) = \|x_t\|_C$, we obtain from (4.3),

$$(4.4)\ \|x_t\|_C \le \|\phi\|_C + [\ 1 + \int_{t_0}^{t} 2L_2(s)\exp\{\int_{t_0}^{s} (2L_2(\tau) + L_1(\tau))\,d\tau\}\,ds$$

From (4.4) and (2.5), If follows that the solution $x(t)$ of the system (1.1) on $[t_o, \infty]$ is bounded. Consequently, the solution $x(t)$ of the system (1.1) is bounded on $[t_0 - r, \infty]$ and the proof of the Theorem 3 is complete.

Remark. It is to observe that our result proved in Theorem 3 yields the stability of the solution $x(t)$ of (1.1) if $\|\phi\|_C$ is small enough.

If $t \ge t_0$ then from (4.1) and making use of (2.6), (2.7), we obtain

$$(4.5)\ \|x(t)\| \le \|\phi\|_C + \int_{t_0}^{t} e^{-\gamma s} L_2(s)[\|x\| + \|x_s\|_C]\,ds + \int_{t_0}^{t} e^{-\gamma s} L_2(s)\int_{t_0}^{s} e^{-\gamma(\tau - s)} L_1(\tau)\,[\|x\| + \|x_\tau\|_C]\,d\tau\,ds$$

$$\le \|\phi\|_C + \int_{t_0}^{t} 2L_2(s)\,e^{-\gamma s}\,\|x_s\|_C\,ds + \int_{t_0}^{t} 2L_2(s)\int_{t_0}^{s} L_1(\tau)\,e^{-\gamma t}\,\|x_\tau\|_C\,d\tau\,ds$$

From (4.5) and by proceeding as in the proof of Theorem 2, we get

$$(4.6)\ \|x_t\|_C \leq \|\phi\|_C + \int_{t_0}^{t} 2L_2(s)\, e^{-\gamma s} \|x_s\|_C\, ds$$

$$+ \int_{t_0}^{t} 2L_2(s) \int_{t_0}^{s} L_1(\tau)\, e^{-\gamma\tau} + \|x_\tau\|_C\, d\tau\, ds$$

Now, since $\gamma > 0$ and $t \geq t_0 \geq 0$, we have $e^{-\gamma t}$ 5 $e^{-\gamma t}$5 1. Multiplying L.H.S of (4.6) by $e^{-\gamma t}$ and R.H.S. of (4.6) by $e^{-\gamma t 0}$, we obtain

$$(4.7)\ e^{-\gamma t}\|x_t\|_c \leq \|\phi\|_C\, e^{-\gamma t_0} + \int_{t_0}^{t} 2L_2(s)\, e^{-\gamma t_0}\, e^{-\gamma s}\|x_s\|_c\, ds$$

$$+ \int_{t_0}^{t} 2L_2(s) e^{-\gamma t 0} \int_{t_0}^{s} L_1(\tau) e^{-\gamma\tau}\, \|x_\tau\|_C\, dr\, ds$$

Applying Lemma 2 with $u(t) = e^{-\gamma t}\|x_t\|_C$, the inequality (4.7) yields

$$(4.8)\ \|x_t\|_C \leq \|\phi\|_C\, [1+\int_{t_0}^{t} 2L_2(s)e\text{-}^{\gamma t_0} \exp(\int_{t_0}^{s} \{2L_2(\tau)e^{-\gamma t_0} + L_1(\tau)\}\, d\tau)\, ds]$$

$$.e^{l(t-t_0)}$$

From (4.8) and (2.5), it follows that

$$\|x_t\|_C \geq Q\, \|\phi\|_C\, e^{\gamma(t-t)}, \qquad t \leq t_0$$

where $Q > 0$ is a constant. Hence, the solution $x(t)$ of system (1.1) is uniformly slowly growing and the proof of the Theorem 4 is complete.

ACKNOWLEDGEMENT

The author expresses his sincere gratidue to Professor B.G. Pachpatte for many helpful discussions and suggestions during the preparation of this paper. This research work is supported by Marathwada University, Aurangabad (Maharashtra) INDIA.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp-139-142

ON ORTHOGONAL IDEALS IN GROUPRINGS

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(Received 15.9.92)

ABSTRACT

In this paper we define orthogonal ideals of a groupring and we prove when the ring is Z_p, p - a prime and G - a cyclic group of order n, there exists a pair of orthogonal ideals I_1 and I_2 in Z_pG such that $Z_p G = I_1 \cup I_2 \cup S$ where S is a semigroup under multiplication. Further we prove that these ideals are cyclic codes of dimension n-1 and 1 respectively.

DEFINITION 1

Let R be a ring. We say two ideals I and J are orthogonal if $I . J = O$ where $I. J = \{ i.j : i \in I, j \in J \}$.

The same definition holds for any grouping. Trivially every ideal is orthogonal to the (O) ideal.

Proposition 2

If R is a ring in which $R^2 = 0$, then every pair of ideals of R is orthogonal.

Proof: Obvious.

Proposition 3

If R is a ring in which $R^2 = 0$ and G is any group then in the grouping RG every pair of ideals is orthogonal.

Proof

As $R^2 = 0$ so $1 \notin R$. We have $RG.RG = (RG)^2 = RG = 0$. Hence by proposition 2, every pair of ideals in RG is orthogonal.

Example 1

$Z_2 = (0,1)$ and $G = \langle g : g^2 = 1 \rangle$. Then $Z_2G = (0, 1, g, 1+g)$. Let $I_1 = (0, 1+g)$, $I_2 = (0, 1+g)$,

$S = (0,1,g)$. Here I_1 and I_2 are orthogonal ideals and S is a semigroup under

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multiplication. Further

$$Z_2G = I_1 \cup I_2 \cup S$$

Example 2

Let $Z_2 = \{0,1\}$, $G = \{ g : g^3 = 1\}$ Then

$Z_2G = \{0, 1, g, g^2, 1+g, 1+g^2, g+g^2, 1+g+g^2\}$

Let $I_1 = \{0, 1+g, 1+g^2, g+g^2\}$, $I_2 = \{0,1+g+g_2\}$

Clearly $I_1 . I_2 = 0$. Further $Z_2G = I_1 \cup I_2 \cup S$ where $S = \{0, 1, g, g^2, 1+g+g^2\}$ is a semigroup under multiplication.

Example 3

$Z_2 = \{0,1\}$, $G = \{g : g^4=1\}$

$Z_2G = \{0, 1, g, g^2, g^3, 1+g, 1+g^2, 1+g^3, g+g^2, 1+g+g^2, g+g^3, 1+g+g^3, 1+g^2+g^3, g+g^2+g^3, 1+g+g^2+g^3\}$

Let $I_1 = \{ 0, 1+g, 1+g^2, 1+g^3, g+g^2, g+g^3, g^2+g^3, 1+g+g^2+g^3 \}$

$I_2 = \{ 0, 1+g+g^2+g^3 \}$

Clearly I_1 and I_2 are orthogonal. Now $Z_2G = I_1 \cup I_2 \cup S$ where $S = \{0, 1, g, g^2, g^3, 1+g+g^2, 1+g+g^3, 1+g^2+g^3, g+g^2+g^3 \}$ is a semigroup under multiplication.

Example 4:

$Z_3 = \{ 0, 1, 2 \}$, $G = \{ g : g^2 = 1\}$

$Z_3G = \{ 0, 1, 2, g, 2g, 1+g, 2+g, 1+2g, 2+2g\} = I_1 \cup I_2 \cup S$, where $I_1 = \{ 0, 2+g, 2g+1\}$, $I_2 = \{0, 1+g, 2+2g\}$ and $S = \{0, 1, 2, g, 2g, 1+g, 2+2g\}$

Theorem 1:

Let $Z_p = \{ 0, 1, 2, \ldots, p-1\}$ and $G = \{ g : g^n = 1\}$.

Then in the groupring Z_pG there exists a pair of orthogonal ideals I_1 and I_2 such that $Z_pG = I_1 \cup I_2 \cup S$ where S is a semigroup under multiplication. Further $I_1 \cap S = \{0\}$. $I_2 \cap S = \{0\}$ if $p | n$ and $I_2 \subset S$ if $p \nmid n$.

Proof

$$Z_pG := \{\alpha = a_0 + a_1 g + \ldots + a_{n-1} g^{n-1} : a_i \in Z_p\}$$

Let $$I_1 = \left\{ \alpha \in Z_pG : \sum_{1}^{n-1} a_i = 0 \pmod{p} \right\}$$

$I_2 = \{ k\,(g^{n-1} + g^{n-2} + \ldots + g + 1),\ 0 \leq k \leq p\text{-}1\}$ and

$$S = \{0\} \cup \left\{ \alpha \in Z_p G : \sum_{i=1}^{n-1} a_i = 1 \text{ or } 2 \text{ or} \ldots \text{ or } (p-1)\ (\text{mod } p) \right\}.$$

Clearly I_1 and I_2 are ideals in Z_pG. Now suppose $\alpha \in I_1$ and $\beta \in I_2$.
Then $\alpha = a_0 + a_1 g + \ldots + a_{n-1}\, g^{n-1}$, $\sum a_i = 0$ (mod p).
$\beta = k\,(g^{n-1} + g^{n-2} + \ldots + g+1),\ 0 \leq k \leq p\text{-}1.$
Then $\alpha.\beta = (a_0 + a_1 + \ldots + a_{n-1}).\ (k\,(g^{n-1} + g^{n-2} + \ldots + g+1))$
$= \sum a_i.\ (k\,(g^{n-1} + g^{n-2} + \ldots + g+1))$
$=0$, since $\sum a_i = 0$ (mod p).

Hence $I_1.I_2 = \{0\}$. Therefore I_1 and I_2 are orthogonal.

Clearly S is a semigroup under multiplication. Also $Z_pG = I_1 \cup I_2 \cup S$. By the definition of I_1 and S we have

$I_1 \cap S = \{0\}$. Now if $p|n$, then $n = mp$, for some positive integer 'm'. Then $I_2 = \{k(1 + g + \ldots + g^{mp-1},\ 0 \leq k \leq p\text{-}1\}$. If $\alpha \in I_2$, we have
$\sum a_i = k..\ mp = 0$ (mod p). Therefore, $I_2 \subseteq I_1$. Hence $I_2 \cap S = \{0\}$.

If $p \nmid n$, then $p \nmid k.\,n$ for any k such that $1 \leq k \leq p\text{-}1$. Therefore, $\sum a_i \neq 0$ (mod p) for any $\alpha \in I_2$. That is, $\sum a_i = 1$ or 2 or ... or $(p\text{-}1)$ (mod p). Thus $\alpha \in S$ which implies $I_2 \subseteq S$.

Theorem 2:

In Z_pG, where $G = \{ g : g^n = 1\}$, the ideal defined by $I_1 = \{ \sum_{i=0}^{n-1} a_i g^i : \sum a_i = 0$ (mod p) $\}$ is a cyclic code generated by "g-1".

Recall that a cyclic code of length n and dimension k is a principal ideal generated by a divisor of degree n-k of the polynomial g^n-1 in Z_pG.

Proof

Suppose $a_0 + a_1 g + \ldots + a_{n-1}\, g^{n-1} \in Z_pG$. Then we have,
$(g\text{-}1).(a_0 + a_1 g + \ldots + a_{n-1}\, g^{n-1})$
$= a_0 g + a_1 g^2 + \ldots + a_{n-1}\, g^n - a_0 - a_1 g - \ldots - a_{n-1}\, g^{n-1}$
$= (a_{n-1} - a_0) + (a_0 - a_1)g + \ldots + (a_{n-2} - a_{n-1})g^{n-1}.$
Here $\sum a_i = 0$. Hence the above element belongs to I_1.

Conversely, suppose $\sum_{i=0}^{n-1} b_i g^i \in I_1$. This implies that

$$\sum_{i=0}^{n-1} b_i = 0 \ (mod\, p)$$

We will find out an element $\sum_{i=0}^{n-2} a_i g^i$ of $Z_p G$ such that

$$\sum_{i=0}^{n-1} b_i g^i = (g-1).(\sum_{i=0}^{n-2} a_i g^i)$$

For, consider the element $\sum_{i=0}^{n-2} a_i g^i$ where a_i's are given by

$a_0 = -b_0$, $a_1 = -(b_0 + b_1)$, ...,$a_{n-3} = -(b_0 + b_1 + b_{n-3})$,
$a_{n-2} = b_{n-1}$

Now, $(g-1).(\sum_{i=0}^{n-2} a_i g^i) = (g-1).\{-b_0 - (b_0 + b_1)g - ... + b_{n-1} g^{n-2}\}$.

$= b_0 + b_1 g + ... + b_{n-2} g^{n-2} + b_{n-1} g^{n-1}$,

since $\Sigma b_i = 0$ (mode p).

$$= \sum_{i=0}^{n-1} b_i g^i$$

Hence every element of I_1 is a multiple of $(g-1)$ and every multiple of $(g-1)$ belongs to I_1. This implies that I_1 is a cyclic code generated by $(g-1)$ and has dimension $n-1$.

Note : The ideal I_2 generated by $(g^{n-1} + g^{n-2} + ... + g + 1)$ is clearly the cyclic code of dimension 1.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp. 143-146

ON TRILATERAL GENERATING FUNCTIONS OF MODIFIED LAGUERRE POLYNOMIALS

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(Received 14-11-1992)

ABSTRACT:

In this paper we derive a theorem on mixed trilateral generating functions of modified Laguerre polynomials by group theoretic method. A particular case of interest is also pointed out.

AMS classification code (1980) : 33A65.

Keywords and Phrases: Generating functions. Modified Laguerre Polynomials.

INTRODUCTION.

The modified Laguerre polynomial $f^{\beta}_{n}(x)$ is defined by [1]

$$f^{\beta}_{n}(x) = (-1)^n L^{-n-\beta}_{n}(x) = \frac{(\beta)_n}{n!}\, {}_1F_1\left[\begin{matrix} -n; \\ 1-\beta-n; \end{matrix}\, x\right] \tag{1.1}$$

In this note we prove the following theorem in connection with the trilateral generating relations of modified Laguerre polynomials from the Lie group view point.

Theorem : If there exists a bilateral generating relation of the form

$$G(x,u,t) = \sum_{n=0}^{\infty} a_n f^{\beta}_{n+k}(x)\, g_n(u)\, t^n \tag{1.2}$$

where k is a non-negative integer and $g_n(u)$ is an arbitrary polynomial of degree n, then

$$\sum_{n=0}^{\infty} \psi_n(x,u,t)\, z^n = (1-z)^{\beta-1} \exp\left(\frac{-xz}{1-z}\right) G\left(\frac{x}{1-z}, u, tz\right) \tag{1.3}$$

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where

$$(1.4) \qquad \psi_n(x,u,t) = \sum_{p=0}^{n} \binom{n+k}{p+k} a_p (-1)^{n-p} f_{n+k}^{\beta-(n-p)}(x)\, g_p(u)\, t^p$$

PROOF OF THE THEOREM

We shall first define the following linear partial differential operator for modified Laguerre polynomials.

$$(2.1) \qquad R = xyz^{-1} \frac{\partial}{\partial x} - y\frac{\partial}{\partial z} + yz^{-1}(1-x)$$

such that

$$(2.2) \qquad R\left(f^{\beta}_{n+k}(x)\, y^n z^{\beta}\right) = -(n+k+1)\, f^{\beta-1}_{n+k+1}(x)\, y^{n+1} z^{\beta-1}.$$

The extended form of the group generated by R is given by

$$(2.3) \qquad e^{wR} f(x,y,z) = \left(\frac{z}{z-wy}\right) exp\left(\frac{wxy}{z-wy}\right) f\left(\frac{wz}{z-wy}.\, y,\, z-wy\right)$$

Consider the following formula

$$(2.4) \qquad G(x,u,t) = \sum_{n=0}^{\infty} a_n f^{\beta}_{n+k}(x)\, g_n(u)\, t^n$$

where $g_n(u)$ is an arbitrary polynomial of degree n.
Replacing t by tz and then multiplying both sides of (2.4) by y^{β}, we get

$$(2.5) \qquad y^{\beta} G(x,u,tz) = \sum_{n=0}^{\infty} a_n \left(f^{\beta}_{n+k}(x)\, z^n y^{\beta}\right) g_n(u)\, t^n$$

Operating both sides of (2.5) by $exp(wR)$, we get

$$(2.6) \qquad exp\,(wR)\,(y^{\beta} G(x,u,tz))$$

$$= exp(wR) \sum_{n=0}^{\infty} a_n\, f^{\beta}_{n+k}(x)\, y^{\beta} g_n(u)\, (tz)^n$$

The left member of (2.6) becomes

$$\text{(2.7)}\quad exp\left(\frac{-\frac{wxz}{y}}{1-\frac{wz}{y}}\right) y^{\beta}\left(1-\frac{wz}{y}\right)^{\beta-1} G\left(\frac{x}{1-\frac{wz}{y}}, u, tz\right)$$

Also the right member of (2.6) becomes

$$\text{(2.8)}\quad \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{w^p}{p!} R^p \left(f_{n=k}^{\beta}(x)\, y^{\beta} z^n\right) g_n(u)\, t^n$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{(-w)^p}{p!}(n+k+1)_p\, f_{n+k+p}^{\beta-p}(x)\, z^{n+p}\, y^{\beta-p}\, g_n(u)\, t^n$$

$$= \sum_{p=0}^{\infty} z^n \sum_{p=0}^{\infty}\binom{n+k}{p} a_{n-p}\left(-\frac{w}{y}\right)^p f_{n+k}^{\beta-p}(x)\, y^{\beta}\, g_{n-p}(u)\, t^{n-p}$$

Then equating (2.7) and (2.8) we get

$$\text{(2.9)}\quad \left(1-\frac{wz}{y}\right)^{\beta-1} exp\left(\frac{-\frac{wxz}{y}}{1-\frac{wz}{y}}\right) G\left(\frac{x}{1-\frac{wz}{y}}, u, tz\right)$$

$$= \sum_{n=0}^{\infty} \psi_n(x, u, t)\, z^n$$

where

$$\text{(2.10)}\quad \psi_n(x, u, t) = \sum_{p=0}^{n}\binom{n+k}{p+k} a_p\left(-\frac{w}{y}\right)^{n-p} f_{n+k}^{\beta-(n-p)}(x)\, g_p(u)\, t^p$$

putting w/y =1 in (2.9), we get

$$\text{(2.11)}\quad (1-z)^{\beta-1} exp\left(\frac{-xz}{1-z}\right) G\left(\frac{x}{1-z}, u, tz\right) = \sum_{n=0}^{\infty} \psi_n(x, u, t)\, z^n$$

where

$$(2.12)\quad \psi_n(x,u,t) = \sum_{p=0}^{n} \binom{n+k}{p+k} a_p (-1)^{n-p} f_{n-k}^{\beta-(n-p)}(x)\, g_p(u)\, t^p$$

This completes the proof of the theorem.

Putting $k=0$, we get the following result of modified Laguerre polynomeals:

Corollary: If there exists a bilateral generating relation of the form

$$(2.13)\qquad G(x,y,t) = \sum_{n=0} a_n f_n^{\beta}(x)\, g_n(u)\, t^n$$

then

$$(2.14)\qquad (1-z)^{\beta-1} \exp\left(\frac{-xz}{1-z}\right) G\left(\frac{x}{1-z}, u, tz\right) = \sum_{n=0}^{\infty} \psi_n(x,u,t) z^n$$

$$(2.15)\qquad \psi_n(x,u,t) = \sum_{p=0}^{n} \binom{n}{p} a_p (-1)^{n-p} f_n^{\beta-(n-p)}(x)\, g_p(u)\, t^p$$

ACKNOWLEDGEMENT

The anthor is grateful to Dr. A.K. CHONGDAR, Reader, Department of Mathematics, Bangabasi Evening College, Calcutta for his kind help and gridance in the procpration of this paper.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp-147-152

ROBUST ROW-COLUMN DESIGNS WITH RANDOM ROW AND COLUMN EFFECTS

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(Received : 30-03-1991, after revision: 23-12-1992)

ABSTRACT

Gopalan and Dey [4] have developed a criterion, on the lines similar to one suggested by Box and Draper [2], for characterising the robust experimental designs assuming a fixed effect model or model - I (Eisenhart [3]). In this paper a criterion for characterising robust experimental row-column designs assuming random row and column effects has been developed and the robustness of the row column designs with random row and column effects has been examined.

Mathematics Subject Classification (1991) : *62K99*

Keywords : Random effect model, robustness, idempotent matrices, eigen values, outlier, equi-replicate, Kronecker product of matrices, heterogeneity, dispersion matrix, normal equations, incidence matrix.

INTRODUCTION

It is assumed that the analysis of variance model for the observations can be approximated by linear combinations (functions) of certain unobservable quantities known as 'effects'. The 'effects' not directly observable quantities are more of less idealised formulations of some properties of interest to the investigator in the phenomenon underlying the observations. The purpose of the analysis is to make inferences about some of the effects and these inferences to be valid regardless of the magnitudes of certain other effects, which may be present in the linear combinations. Each affect is regarded as either an unknown constant or else as a random variable. Following Eisenhart [3], if the effect is treated as an unknown constant, it is called a fixed effect model or model -I, otherwise it is called a random effect model or model - II. The equation expressing the observations as linear combinations of the effects is known as model equation. It is assumed that in every model equation there is at least one set of random effects equal in number to the number of observations, a different one

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of which appears in every observation. This is called the residual effect or the error term. Furthermore, there is usually one fixed effect, which appears in every model equation and is known as the general constant and is the mean in some sense of the observations.

A number of criteria have been enumerated by Box [1] for judging a good response surface design for fitting an empirical interpolation function $Y = X\beta + e$ where $Y = (y_1, y_2,..., y_n)'$ is an nx 1 vector of response observations, X is an nxp matrix of predictor variable observations, β is a px1 vector of parameters to be estimated and e is an nx 1 vector of residuals. On of the criterion is that the design should be insensitive to the wild observations. Box and Draper [2] have derived an appropriate numerical measure of a design's desirability in relation to the insensitivity to wild observations. Box and Draper [2] have shown that in order that the predicted response y at any given point be insensitive to the outlier, the quantity $r = \Sigma r_{uu}^2$ should be minimised, where r_{uu} is the u-th diagonal element of the matrix $R = X(X'X)^{-1}X$. The designs which minimise r are called *robust*.

2. ROW-COLUMN DESIGNS WITH RANDOM ROW AND COLUMN EFFECTS

We shall consider a row-column design in which v treatments are arranged in p rows and q columns, i-th treatment being replicated r_i times. The model assumed in this case will be

$$Y = \mu J + \pi \tau + L \phi + M \Phi + e$$

where $y = ((y_{ijk}))$ is the vector of response of the i-th treatment in j-th row and k-th column, μ is the general mean effect, $\tau = (\tau_1. ..., \tau_v)'$ is the vector of treatment effects, $\phi = (\phi_1 ...,\phi_p)$ is the vector of row effects, $\Phi = (\phi,...,\phi_q)'$ is the vector of column effects, e is the vector of random errors, independently normally distributed with mean vector O and dispersion $\sigma^2 I$. Here ϕ and Φ are assumed to be vector of random effects distributed normally with mean vectors O and dispersion matrices $\sigma_r^2 I$ and $\sigma_c^2 I$ respectively. π', L' and M' are the incidence matrices of treatments versus observations, row versus observations and column versus observations respectively. I is an identity matrix and J is a vector with all elements as unity. The above model may be written as

$$Y = X\,0 + e_1$$

where $e_1 = L\phi + M\phi + e$, $X = [J : \pi_{n,v}]$ and $\theta = (\mu, \tau)'$ we have the dispersion matrix of e_1 as,

$$\begin{aligned} D(e_1) &= LL'\,\sigma_r^2 + MM'\,\sigma_c^2 + I\,\sigma^2 \\ &= Z^{-1}\,\sigma^2 \quad \text{(say)} \end{aligned}$$

where Z^{-1} is the variance-covariance matrix of the random components and is function of σ_r^2/σ^2 and σ_C^2/σ^2 which are assumed to be fixed and known. Application of usual least square technique to estimate the vector of parameters yields the normal equation,

$$\hat{\theta} = (X'Z^{-1}X)^{-}X\,Z^{-1}Y$$

where $(X'Z^{-1}X)^{-}$ is generalised inverse (g - inverse) of $X'Z^{-1}X$ satisfying $X'Z^{-1}X(X'Z^{-1}X)^{-}X'Z^{-1}X = X'Z^{-1}X$. It is well known that in the absence of any outlier $R_0^2/(n-m)$ is an unbiased estimator of σ^2 where,

$$R_0^2 = Y'\,[Z^{-1} - Z^{-1}X\,(X'Z^{-1}X)^{-}X'Z^{-1}]\,Y$$

m = rank (X) and n is the number of observations.

When u-th observation has added to it a quantity c making it an outlier, the bias or discrepancy in estimating σ^2 through R_0^{*2} is $d_u = c^2a_{uu}/(n-m)$ where R_0^{*2} is the residual sum of squares in the presence of the outlier and a_{uu} is the u-th diagonal element of the matrix

$$\begin{aligned} A &= Z^{-1} - Z^{-1}X\,(X'Z^{-1}X)^{-}X'Z^{-1} \\ &= Z^{-1} - D\,(Z^{-1}X'\,\hat{\theta}) \end{aligned}$$

where D stands for the dispersion matrix. If it were equally likely that c could occur with any of the n observations, giving rise to $d_1,\ldots,d_n$, the average discrepancy would be

$$\bar{d} = c^2 \sum_u a_{uu} \,/\, n(n-m)$$

which obviously is not fixed, unlike in the case of the fixed effect model, but is design dependent through A. In order that no unduly large discrepancy in the estimator of σ^2 is caused by the outlier, the average discrepancy should be minimised. A design minimising the average discrepancy, in case of a random effect model, may be called *robust*.

We have,

$$D(Z^{-1}X'\hat{\theta})/\sigma^2 = Z^{-1}\pi'\,(\pi Z^{-1}\pi')^{-1}\pi' Z^{-1}$$

For getting the average discrepancy we need

$$\begin{aligned} trace\ Z^{-1}\pi' (\pi Z^{-1}\pi')^{-1}\pi Z^{-1} &= trace\ (\pi Z^{-1}\pi')^{-1}\pi Z^{-1} Z^{-1}\pi' \\ &= trace\ (\pi Z^{-1}\pi')^{-1}\pi Z^{-2}\pi' \end{aligned}$$

We examine below, robustness of row-column designs based on the criterion discussed above.

3. ROBUSTNESS OF ROW-COLUMN DESIGN

In case of a row-column design in v treatments, p rows and q columns the form of the matrix $\mathbf{Z}$ will be,

$$\mathbf{Z} = \begin{bmatrix} (1+\delta_2)+\delta_1 J & \delta_2 I & \ldots & \delta_2 I \\ \delta_2 I & (1+\delta_2)+\delta_1 J & \ldots & \delta_2 I \\ \vdots & \vdots & & \\ \delta_2 I & \delta_2 I & \ldots & (1+\delta_2)+\delta_1 J \end{bmatrix}$$

or $Z = I_p \otimes [I_q + \delta_1 J_q] + J_p \otimes \delta_2\ I_q$

$$= I_p \otimes I_q + \delta_1 I_p \otimes J_q + \delta_2 J_p \otimes I_q$$

where $A \otimes B$ denotes the Kronekcer product of matrices A and B. J_n is an nxn matrix with all elements as unity, $\delta_1 = \sigma_r^2/\sigma^2$ and $\delta_2 = \sigma_c^2/\sigma^2$. The form of Z^{-1} will be

$$Z^{-1} = a_{oo}\, I_p \otimes I_q + a_{01}\, I_p \otimes J_q + a_{10}\, J_p \otimes I_q + a_{11}\, J_r \otimes J_q$$

where ,

$a_{oo} = 1$

$a_{01} = -\ \delta_1\ /(1+q\ \delta_1)$

$a_{10} = -\ \delta_2\ /(1+p\ \delta_2)$

$a_{11} = -\delta_1\delta_2\ (2+p\delta_2\ +\ q\delta_1)/[(1+p\delta_2)\ (1+q\delta_1)\ (1+p\delta_2+q\delta_1)$

In order to compute the average discrepancy in estimating σ^2 through R^{*2}_0 in the presence of an outlier, we need trace $(\pi Z^{-1}\pi')^{-1}\ \pi Z^{-2}\pi'$. For a row-column design we have,

$$\begin{aligned} \pi Z^{-1}\pi' &= \pi\,[\,I_p \otimes I_q + a_{01} I_p \otimes J_q + a_{10}\, J_p \otimes I_q\ +\ a_{11}\, J_p \otimes J_q\,]\ \pi' \\ &= r^d + a_{01} N_1\, N_1' + a_{10} N_2 N_2' + a_{11}\, rr' \end{aligned}$$

and

$$\pi Z^2 \pi' = r^d + b_{01} N_2 N_1' + b_{10} N_2 N_2' + b_{11} rr'$$

where

$$b_{01} = -\delta_1 (2 + q\delta_1) / (1 + q\delta_1)^2$$

$$b_{10} = -\delta_2 (2 + p\delta_1) / (1 + p\delta_2)^2$$

$$b_{11} = 2(1 + p\delta_2 + q\delta_1) - pq(p\delta_2 + q\delta_1)\, \delta_1 \delta_2 / [(1 + p\delta_2 + q\delta_1)(1 + p\delta_2)(1 + q\delta_1)]$$

and r^d is a diagonal matrix with elements of the replication vector r in the diagonal elements, N_1 and N_2 are the treatment versus rows and treatment versus column incidence matrices respectively.

We shall consider binary equi-replicate row-column designs for which $N_1 N_1'$ and $N_2 N_2'$ commute. For such designs

$$\text{trace } \left(\pi Z'\pi'\right)'\left(\pi Z^2 \pi'\right) = \frac{r + b_{01} vr + b_{10} vr + b_{11} vr^2}{r + a_{01} vr + a_{10} vr + a_{11} vr^2} + \sum_{i=1}^{v-1} \frac{r + b_{01}\theta_{1i} + b_{10}\theta_{2i}}{r + a_{01}\theta_{1i} + a_{10}\theta_{2i}}$$

where θ_{1i} and θ_{2i} are the *i-th* eigen values of the matrices $N_1 N_1'$ and $N_2 N_2'$ respectively. From equation (3.1) it is clear that the average discrepancy is not fixed, as in the case of fixed effect model, but is design dependent through θ_i's. Inorder that the average discrepancy is minimum, the trace given in (3.1) should be maximum subject to the condition that sum of all the eign values is constant.

So maximising (3.1) subject to the condition that $\sum_{i=1}^{v-1} \theta_{1i} = r(v-k)$ and $\sum_{i=1}^{v-1} \theta_{2i} = r(v-k)$, we may see that all θ_{1i}'s are same and so are θ_{2i}s. Let $\theta_{1i} = \theta_1$ and $\theta_{2i} = \theta_2$ so that $\theta_1 = r(v-q)/(v-1)$ and $\theta_2 = r(v-p)/(v-1)$

The characteristic vector corresponding to the eigen value rq is $rq\, JJ'$ and the eigen vector corresponding to the eigen value $r(v-q)/(v-1)$ is $(I - JJ'/v)$, so we have,

similarly,
giving,

$$N_1 N_1' = rq\, JJ'/v + r(v-q)(v-1)^{-1}(I - JJ'/v)$$

$$N_2 N_2' = rp\, JJ'/v + r(v-p)(v-1)^{-1}(I - JJ'/v)$$

$$pN_1 N_1' + qN_2 N_2' = r(v-1)^{-1}[\{v(p+q) - 2pq\} I - \{p + q - 2pq\} JJ']$$

which is satisfied by balanced incomplete row-column designs (Singh [5]), showing that balanced incomplete row-column designs with random row and column effects are robust.

ACKNOWLEDGEMENT

The author is grateful to the referee for his constructive comments on the earlier draft of the paper.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 153-158

INTER BLOCK ANALYSIS OF BALANCED INCOMPLETE BLOCK DESIGNS WITH NESTED ROWS AND COLUMNS

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(Received: 25-12-1992, after revision: 30-03-1993)

ABSTRACT

Balanced incomplete block designs with nested rows and columns (BIBRC) were introduced by Singh and Dey [2] as generalisation of the Lattice Square designs. The intra-block analysis of these designs has been developed by Singh and Dey [2]. Since the blocks of the BIBRC are incomplete, assuming the block effects as random and row, column classification complete, an attempt has been made in this paper to develop the inter block analysis of these designs.

Mathematics Subject Classification (1991) : 62K10

Keywords : Lattice Square designs, heterogeneity, orthogonal, dispersion matrix, Kronecker product of matrices, maximum likelihood estimates, random effect model.

INTRODUCTION

The experimental units within a complete block are often subjected to more than one source of variation. Recognising the need for appropriate experimental designs for such situations Yates [3,4] developed a group of block designs with possibility of eliminating heterogeneity in two directions within the complete block. Singh and Dey [2] defined and studied a class of experimental designs which are arrangements of treatments in several sets (blocks) of rows and columns such that within each set (block) the row versus column classification is orthogonal, though the sets themselves are incomplete. These designs may be viewed as the generalization of the Lattice Square

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designs of Yates [4]. Designs defined by Singh and Dey [2] are different from the designs of Preece, Pearce and Kerr [1] who consider designs with three mutually orthogonal and fully crossed classifications while Singh and Dey [2] consider designs with a set of blocks (sets) within which are nested two more classifications, rows and columns.

NOTATIONS AND FORMULATION OF THE MODEL FOR ANALYSIS

Let us suppose that v treatments have been allotted to s blocks (sets) and the experimental units within each set are further arranged in p rows and q columns, enabling to control the variability within a set, in two directions. The model assumed in this case is

$$y_{i_{(j_m j_l)}} = \mu + \Omega_j + \phi_{j_m} + \delta_{jl} + \tau_i + \varepsilon_{i_{(j_m j_l)}} \tag{1}$$

where, $y_{i_{(j_m j_l)}}$ is the response obtained from the unit receiving i-th treatment in the $(j_m j_l)$-th cell of the j-th set, μ is the general mean effect, Ω_j is the effect of the j-th set (block) ($l = 1, \ldots\ldots, s$), ϕ_{j_m} is the effect of m-th row of the j-th set ($m = 1, \ldots\ldots, p$), δ_{jl} is the effect of the l-th column in the j-th set ($l = 1, \ldots\ldots, q$), τ_i is the effect due to i-th treatment ($i = 1, \ldots\ldots, v$) and $\varepsilon_{i_{(j_m j_l)}}$'s are uncorrelated residuals distributed normally with mean zero and constant variance σ^2. Since model (1) is assumed as random effect model, we further assume the following distributional properties of different effects i.e. δ_{jl} is distributed normally with mean zero and variance $\sigma^2_{c_j}$, ϕ_{j_m} is distributed normally with mean zero and variance $\sigma^2_{r_j}$ and Ω_j is distributed normally with mean zero and variance σ_s^2. Also ϕ_{j_m}, δ_{jl}, and $\varepsilon_{i_{(j_m j_l)}}$ are independently distributed. further,

$$V[\, y_{j(m,l)} \,] = \sigma^2 + \sigma^2{}_s + \sigma^2{}_{r_j} + \sigma^2{}_{c_j}$$

$$Cov[\, y_{j(m,l)}, y_{j(m',l)} \,] = \sigma^2{}_s + \sigma^2{}_{c_j} \qquad , m \neq m'$$

$$Cov[\, y_{j(m,l)}, y_{j(m,l')} \,] = \sigma^2{}_s + \sigma^2{}_{r_\partial} \qquad , l \neq l'$$

$$Cov[\, y_{j(m,l)}\, ,\, y_{j(m',l')}\,] = \sigma^2{}_s \qquad , l \neq l' , m \neq m'$$

$$Cov[\, y_{j(m,l)}\, ,\, y_{j(m,l)}\,] = 0 \qquad , j \neq j'$$

Let $y_{j\,(m,.)}$, $y_{j(\,.\,,.)}$ and $y_{.(.\,.\,.)}$ denote the total of the l-th column of the j-th set, total of the j-th set and grand total respectively. Further let us denote the vectors of the column totals of the j-th set, totals of the set (block) and grand total respectively as,

$$Y_{j(m,)} \quad = [\, y_{j(m,1)}\, ,\ \ldots\ldots\ldots\ldots,\ y_{j(m,q)}\,]'_{qx1}$$

$$Y_{j(.)} \quad = [\, y_{j(1,)}\, ,\ldots\ldots\ldots\ldots\ldots,\ y'_{j(m,)}\,]'_{pqx1}$$

$$Y_{(.)} \quad = [\, y'_{1(.)}\ ,\ldots\ldots\ldots\ldots\ldots,\ y'_{s(.)}\,]'_{spqx1}$$

it may easily be verified that

$$D\left[Y_{j(m,)}\right] = (\sigma^2 + \sigma^2{}_{c_j})I + (\sigma^2{}_s + \sigma^2{}_{r_j})J_{qq}$$

$$Cov\left[Y_{j(m,)}, Y_{j(m',.)}\right] = \sigma^2_{cj}\, I + \sigma^2_s\, J_{q,q} \qquad m \neq m'$$

$$Cov\left[Y_{j(m,)}, Y_{j'(m',)}\right] = 0_{q,q} \qquad j \neq j' ,\ m \neq m'$$

where **D**(.) stands for the dispersion matrix, ***I*** is the identity matrix and ***J*** is the matrix with all elements as unity. Thus,

$$D[y_{j(m,)}] = \sigma^2 I_{pq} + \sigma^2_{r_j} I_p \otimes J_{q,q} + \sigma^2_{c_j} J_{p,p} \otimes I_p + J_{qp,\,pq}$$

$V_j\,(say)\ j = 1, \ldots, s$

$$D[Y_{.(.,.)}] = diag(V_1, \ldots, V_s)$$

$$= I_s \otimes (\sigma^2 I_{pq} + \sigma_s{}^2 J_{pq,qp}) + diag\,(\sigma^2_{r_1}, \ldots, \sigma^2_{r_s}) \otimes (I_p \otimes J_{q\text{-}p})$$

$$+\, diag\,(\sigma^2_{c_1}\ , \ldots,\ \sigma^2_{c_s}) \otimes (J_p \otimes I_p) \qquad (2)$$

$$= V\ (say)$$

Where ⊗ stands for Kronecker product of matrics. Under these assumptions, the joint distribution of the elements of the response vector y is given by

$$f(y,\ \theta) = [(2\pi)^{n/2}\,|V|^{1/2}]^{-1}\ exp^{\,-1/2}\ (y-\mu\ J - \Delta'\tau)'\ \ V^{-1}\ (y-\ \mu\ J - \Delta'\tau)$$

Where $\theta' = (\mu, \sigma^2, \sigma^2_s, \sigma^2_{r_j}, \sigma^2_{c_j})$, $\Delta_{v,m}$ is treatment versus observation incidence matrix and $|V|$ is the determinant of V, From (2) we have,

$$V^{-1} = diag\ (V_1^{-1}, V_2^{-1}, \ldots, V_s^{-1})$$

with $\quad V_j^{-1} = a_j I_p \otimes I_q + b_j I_p \otimes J_p + c_j\ J_{p,p} \otimes I_q + d_j J_{p,p} \otimes J_{q,q}$

where a_j, b_j, c_j and d_j are functions of σ^2, $\sigma_s^2\sigma^2_{c_j}$ and $\sigma^2_{r_j}$ given by

$$a_j = 1/\sigma^2$$

$$b_j = -\rho^2_{r_j}\ /\{(1+q\rho^2_{r_j}\ )\ \sigma^2\}$$

$$c_j = -\rho^2_{c_j}\ /\{(1+q\rho^2_{c_j})\ \sigma^2\}$$

$$d_j = -\ \frac{\rho^2_{r_j}\,\rho^2_{c_j}(2 + p\,\rho^2_{c_j} + p\,\rho^2_{r_j} + pq\rho^2_s) - \rho^2_s}{\sigma^2\,(1 + p\,\rho^2_{c_j})\,(1 + \rho^2_{r_j})\,(1 + p\rho^2_{c_j} + q\rho^2_{r_j} + pq\rho^2_s)}$$

where $\rho^2_{r_j} = \sigma^2_{r_j}/\rho^2$, $\rho^2_{c_j} = \sigma^2_{c_j}/\sigma^2$ and $\rho_s^2 = \sigma_s^2/\sigma^2$

Writing model (1) as $y = X\beta$, where β is the vector of parameters, maximum likelihood estimates of the vector of parameters may be obtained by solving the normal equations.

$$\left(\sum_{j=1}^{s} X_j'\ V_j^{-1}\ X_j\right)\beta = \sum_{j=1}^{s} X_j'\ V_j^{-1}\ y_j \qquad (3)$$

The left hand side of equation (3) may be simplified further. Let α_{ij} denote the incidence vector of the i-th treatment in the j-th set. Obviously, we have

$$\alpha'_{ij}\alpha_{ij} = r_j$$
$$\alpha'_{ij}\alpha_{i'j} = 0 \qquad\qquad i \neq i' \text{ and}$$
$$J'\alpha_{ij} = r_{ij}$$

r_{ij} being the number of times the i-th treatment has been replicated in the j-th set, r_j is the vector of replication of treatments in the j-th set and $J=(1,\ 1,\ldots\ldots,1)'$, Model (1) may also be written as,

$y_i = \mu J + \tau +$ error components of sets, rows within set and columns within set and the residual

$$E(y_j) = \left[J_{pq,1} \vdots \mathbf{a}_{ij},\ \ldots,\ \mathbf{a}_{vj} \right] \begin{bmatrix} \mu \\ \tau_1 \\ \vdots \\ \tau_v \end{bmatrix}$$

$$= X_j \beta$$

Using these notations it may easily be verified that the term on the left hand side of equation (3) is

$$\left(\sum_{j=1}^{s} X'_j V^{-1}{}_j X_j \right) = \begin{bmatrix} J'V^{-1}J & \sum_{j=1}^{s} J'V_j^{-1}\alpha_{1j} & \sum_{j=1}^{s} J'V_j^{-1}\alpha_{vj} \\ & \sum_{j=1} \alpha'_{ij} V_j^{-1} \alpha_{ij} & \sum_{j=1}^{s} \alpha'_{ij} V_j^{-1} \alpha_{ij} \end{bmatrix}$$

The general solutionof the normal equation(3) is rather complicated. However, for a particular design and for various values of the ratios, ρ_{r_j}, ρ_{i_j} and ρ_s the inter block estimates of the parameters may be obtained by solving the aforesaid normal equations.

ACKNOWLEDGEMENTS

The author is grateful to Dr. A. Dey and Dr. M. Singh, Professor and Scientist S-1 respectively at Indian Agricultural Statistics Research Institute, New Delhi, for their guidance and suggestions during the preparation of this paper. The author is also thankful to the referee for his very helpful comments on the previous version of the paper.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 159-166

HINDU TRIGONOMETRY[†]

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(Received 17.07.1993)

ABSTRACT

In this paper we attempt to give a historical back ground of trigonometry and some trigonometrical formulae form *siddhanta Siromaṇi*. The conclusion is that the trgonometrical functions travelled from India to Europe via Arab.

Mathematics Subject Classifications (1991). : 01A32

Keywords & phrases : Trigonometrical functions, Hindu sine/cosine/versed sine.

THE ORIGIN OF TRIGONOMETRICAL FUNCTIONS AND THEIR HINDU NAMES

JYĀ

The fundamental term "*jyā*" of trigonometry appears in *Ṛgveda* (6.75.3).

योषे॑व शिङ्क्ते॒ वित॒ता॒धि॒ धन्व॒ञ्ज्या इ॒यं सम॑ने पा॒रय॑न्ती ।
अहि॑रिव भो॒गैः पर्ये॑ति बा॒हुं ज्याया॑ हे॒तिं प॑रि॒बाध॑मानः ॥

"*Jyā*" is the original word of Sanskrit language. The literal meaning of *jya* is the string of the *dhanusa* (bow) (See also [10, p. 197) and [17, p. 24].

"*Jyā*" also appears in *Sūrya Siddhānta*. We quote the following (cf. [7], [9] and [14].

राशिलिप्ताष्टमो भागः प्रथमं ज्यार्धमुच्यते ॥
तत्तद्विभक्तलब्धोनमिश्रितं तद् द्वितीयकम् ॥
आद्येनैवं क्रमात् पिण्डान् भक्त्वा लब्धोनसंयुताः ॥
खण्डकाः स्युश्चतुर्विंशज्ज्यार्धपिण्डाःक्रमादमी ॥

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† Its major part was presented at National Seminar on Science in Ancient India Oct 12-16, 1991, Kumaun University, Nainital

Indeed trigonometry forms the base of *Sūrya Siddhānta*, and it developed in India as the stern daughter of astronomy.

A special name for the function *jyā* which we call the sine also appears in the *Ganita - Pada* of *Āryabhaṭiya* of *Āryabhaṭa* (b. 476 A.D.). However instead of *jyā*, he used *ardhajyā* (half chord). A (Hindu) sine table has also been given by *Āryabhaṭa* (see [14, p. 49]).

The word *jya* also appears in *Pnacasiddhāntikā* of *Varahamihira* (b. 505 A.D.) (See [15]). He also formed a (Hindu) sine table with a diameter of 120.

The term *jyā* formed its way into the works of *Brahmagupta* as *kramajyā* i.e. a straight line or *sinus rectus*.

Now the question arises how this word *jyā* is changed to sine. When this word transferred from India to Arab, it was pronounced *jaib* (see Gingerich [4], Smith [13] and Srinivasiengar [14]). And Arabians used this unrelated word. In fact *jaib* means bosom, breast and bay. When Gherardo of Cremona (1150 A.D.) made his translation from the Arabian literature, he used *sinus* for *jaib*. *Sinus* means bosom, bay, a curve, the fold of the toga about the breast, the land about a gulf and a fold in a land.

The word *kramajyā (jyā)* changed into *karaja* and *kardaja* at the hands of Arabians.It appears in the Bagdad school in the 9th century. In particular, *Al-Khowārizmī* used in the extract, which he made from the *Brahmasphutasiddhanta* of *Brahmagupta*. This work is known *Sindhind*. This word *jyā* also appears in the writings of the Spanish *Arab-ibn-Al-Zargāla* (1050 A.D.).

ABBREVIATION OF *JYĀ* (SINE)

The first mathematician who abbreviated sine into sin was Girard (1626 A.D.), (cf. [13]).

KOTIJYĀ (COSING)

Cosine is the modern name of Sanskrit word *kotijyā* that was used by ancient Indian astronomers. It appears in *Sūrya Siddhānta* and in the works of *Āryabhaṭa*. *Koti* means the (perpendicular) side of a right triangle, but it also means the end of *dhanuṣa* (bow). In other words *koṭijyā* means complementary *jya*

ABBREVIATION FOR *KOṬIJYĀ* (COSINE)

Plato of Tivoli (1120 A.D.) used *chorda residui* or the complement angle Regiomontanus (1463 A.D.) used *sinus rectus* complementi. Rhaeticus (1515 A.D.) preferred basis. Vieta (1579 A.D.) used *sinus residuae*. Magini (1609 A.D.) used *sinus secundes*. Edmud Gunter (1620 A.D.) suggested *co-sinuṣ* John Newton (1658 A.D.) used *cosinus* and this word received general favour. Jonas Moore (1674 A.D.) used cos and this symbol generally has been adopted by later writers (cf. Smith [13]).

UTKRAMAJYĀ (VERSED SINE)

The function *utkramajyā* first appears in *Surya Siddhānta* (280-400 A.D.) and immediately in the writings of *Āryabhata* I. It is also known as *ujjayā*. *Bhāskarācārya* in his *Lilāvati* calls it *Śara* (see below). It is (now) denoted by $Hvers\,Sin\theta (Hvers\,\theta) = R - H\cos\theta$, wherein R is the radius of the circle. In modern language it is denoted by $Versin\theta = 1 - \cos\theta$ (see Smith [13, part II, p. 618] and Kumari [7, Part I, p. 117]) From India it passed to Arab and they called it turned chord that is the arc of a circle.

THE VALUES OF *JYĀ* AND *KOṬIJYĀ* FUNCTIONS

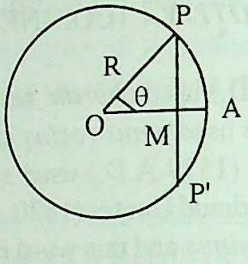
Munjāla (932 A.D.) defined the positive and negative values of the functions *jyā* and *Kotijyā* (see for instance, [16, p. 250]), in the following verse of his *Laghumānasa*

ग्रह: स्वोच्चोनित: केन्द्र तदूध्वाधोऽर्धजो भुज: ।
धनर्ण पदश: कोटी घ्नर्णर्ण धनात्मिका ।।

This means :

In the upper semicircle (1-2 quadrants), the value of the function *jyā* remains positive and the function *kotijyā* remains +,-,-,+ in the four quadrants respectively.

HINDU DEFINITION OF *JYĀ* (SINE), *KOṬIJYĀ* (COSINE) AND *UTKRAMAJYĀ* OR *ŚARA* (VESED SINE)



In figure 1, 0 is the centre of the circle, PP' is the full chord of the circular arc PAP' and A is the middle point of the arc PAP'. On account of its shape PAP' is called the *Cāpa* . Let the arc PA subtend an angle θ at the cente 0. In Hindu trigonometry the angle θ is connoted by the corresponding arc AP of the circle. The chord PP' is the string of the bow and is called *jyā* in Sanskrit. PM which is half of the full chord PP' is known as (*Ardhyajyā* or simply *jyā*). Thus

Jya $\overset{\wedge}{PA} = PM = H\sin\theta$

Kotijya $\overset{\wedge}{PA} = OM = H\cos\theta$

Versed sine $\overset{\wedge}{PA} = MA = H\text{vers}\,\theta$

where PA stands for the arc PA and H for Hindu. Thus

$H\sin\theta = R\sin\theta$, $H\cos\theta = R\cos\theta$ and
$H\text{vers}\,\theta = R\text{vers}\,\theta = R - H\cos\theta$ (see [1],[2],[5] and [8]).

VALUE OF RADIUS R

The circumference of the circle is divided into 360 x 60 = 21600 equal parts and each part is taken to be a unit length, i.e. *Kalā* (minute).

Thus $2\pi R = 21600$
$R = 21600 \div 2\pi$, where $\pi = 3.1416$

The radius R is generally supposed to be 3438' (see [1], [2], [3], [7], [11], [12] and [16]. Recently, Hogendijk [6] has illustrated that R is half the circumference of the circle, and that

$R = (180/\pi)^\circ = (10800/\pi)' = 3437.7' \cong 3438'$

He [op. cit.] further says, "Sine tables with the base $R = 3438'$ are of (pre-Islamic) Indian origin and were known in the early Islamic tradition, although no such tables survive in the Arabic sources".

Sometimes the radius is taken to be 120 units, and *Brahmagupta's Laghujyā* is indeed Hsine in this case. *Bhāskarācārya* also uses these *Laghujyas* in the rectification of planets. *Bhāskarācārya takes* the $R = 3437.4766$ units (see [1] which is evidently a slightly better than 3438 units.

5. SOME BASIC FORMULAE

It is obvious from the right triangle POM,

(1) $Hsin^2\theta + Hcos^2\theta = R^2$

In modern language,

$$R^2 sin^2\theta + R^2 Cos^2\theta = R^2$$

which gives the well - known formula of trigonometry :

$$sin^2\theta + cos^2\theta = 1$$

Bhāskarācārya writes in *Golādhyāya* ;

> क्रमोत्क्रमज्याकृतियोगमूलाद्दलं तदर्धांशकशिञ्जिनी स्यात् ।
> त्रिज्योत्क्रमज्यानिहतेर्दलस्य मूलं तदर्धांशकशिञ्जिनी वा ॥
> तस्याः पुनस्तद्दलभागकानां कोटेश्च कोट्यंशदंलस्य चैवम् ।
> अन्यज्यकासाधनमुक्तमेवं पूर्वैः प्रवक्ष्येऽथ विशिष्टमस्मात् ॥

Thus according to Somayaji [2] means :

(2) $Hsin\,\theta/2 = \frac{1}{2}\,(Hsin^2\theta + Hvers^2\,\theta)^{1/2}$

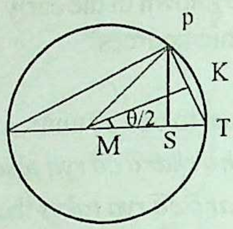
(3) $Hsin\,\theta/2 = (\frac{1}{2}\,R \times Hvers\ \theta)^{1/2}$

We give a proof of these formulae,

Proof. In figure 2, let $\angle PMT = \theta$, PS $\perp$ OT, K the middle point of PT and R

the radius of the circle OPT. Then

$$KMT = \frac{\theta}{2} \text{ and } TK = \frac{1}{2}\, PT$$



The following steps are evident from the right triangle PST.

$$PT^2 = PS^2 + ST^2$$

$$PT^2 = Hsin^2\theta + Hvers^2\theta$$

$$\text{and } Hsin\,\frac{\theta}{2} = TK = \frac{1}{2}PT = \frac{1}{2}(Hsin^2\theta + Hvers^2\theta)^{1/2}$$

From the right triangle POT,

$$PT^2 = TS \times OT$$

$$PT^2 = (Hvers\,\theta) \times 2R \text{ (since } OT = 2R)$$

$$PT = (2R \times Hvers\,\theta)^{1/2}$$

$$2Hsin\,\frac{\theta}{2} = (2R \times Hvers\,\theta)^{1/2}$$

$$Hsin\,\frac{\theta}{2} = \frac{1}{2}(2R \times Hversa\,\theta)^{1/2} = (\frac{1}{2}\,R \times Hvers\theta)^{1/2}$$

Bhāskarācārya has given several other trigonometrical formulae. The verse numbers (12), (13), (14), (15), (16), (17), (18), (19), (20), (21) and (22) of *Golādhyāya* (*jyot-pattivāsanā*) give the following formulae :

(4) $Hsin\dfrac{(90^\circ + x)}{2} = [1/2(R^2 + RHsin\,x)]^{1/2}$

5) $\sqrt{2}\,Hsin\left(\dfrac{x - y}{2}\right) = [1/2\{(Hsinx - Hsiny)^2 + (Hcosx - Hcosy)^2\}]^{1/2}$

(6) $[(Hcosx - Hsinx)^2]^{1/2} = Hsin(45^\circ - x)$

(7) $R - (2Hsin^2x)/R = Hsin(90^\circ - 2x)$

(8) $Hsin(x \pm 1) = Hsinx\ (1 - 1/6569) \pm 10/573\ Hcosx$

(9) $H_{r+1} = H_r(1 - 1/476) + Hcosx_r\ (100/1529)$

(10) $Hsin(x + y)\ (Hsinx.\ Hcosy + Hcosx.Hsiny) \div R$

For details refer to Somayaji [1] and [2].

For the proof of the formula (8) Somayaji has suggested the interpolation method. We give below a somewhat straight proof.

To prove (8), we note that

$Hsin(x+1)^\circ = (Hsinx^\circ.Hcos1^\circ)/R \pm (Hsin1^\circ.Hcosx^\circ)/R$

Now, put $Hsin1^\circ = 60'$, $Hcos1^\circ = 3437.4766$ and

$R = 3438'$ to get,

$$\begin{aligned} Hsin(x\pm1)^\circ &= (Hsinx^\circ.3437.4766/3438) \pm (60Hcosx^\circ/3438) \\ &= Hsinx\ (1 - .5234/3438) \pm (10Hcosx/573) \\ &= Hsinx(1 - 1/6568.5899) \pm (10/573)\ Hcosx \\ &= Hsinx\ (1 - 1/6569) \pm (10\ Hcosx)/573 \end{aligned}$$

For the solution of diurnal problems the verse numbers (54), (55) and (57) of *Grhaganita* (*Spastādhikāra*) give the following formula :

(11) $Hsin\ \alpha = R(Hsin^2\lambda - Hsin^2\ \delta)^2 + Hcos\ \delta$

(12) $Hsin\ \alpha = Hsin\ \lambda \times Hcos\omega \div Hcos\ \delta$

Wherein λ, δ, α and ω are the longitude declination, right ascensian of the sun and obliquity of the ecliptic respectively. For details refer to Somayaji [1] and [2]

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp.167-174

ON CONVERGENCE AND FIXED POINTS OF SEQUENCES OF MAPPINGS

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(Received 29.01,1992 and in revised form 02.12.1994)

ABSTRACT

Two results on common fixed points have been obtained. The first theorem deals with convergence of the sequence of common fixed points of four sequences of selfmappings on a metric space. In the second result an existence theorem for common fixed points of two sequences of mappings with two selfmappings has been established.

Mathematics subject classification (1991) : 54H25, 47H10

Key words and phrases : Weakly commuting mappings, sequences of mappings, common fixed point

INTRODUCTION

Several authors have dealt on the question of convergence of a sequence of fixed points or commom fixed points of mappings and many interesting results have been established. In the present paper we prove a theorem which gives sufficient conditions under which the uniform convergence of four sequences of mappings to four given mappings satisfying a Meir and Keeler type contractive condition implies the convergence of the common fixed points of the sequences of mappings to a common fixed point of the four mappings. The second theorem is an extension, to the case involving sequences of mappings, of the following recent theorem by the first author [8,p.78], [9]:

Theorem A Let P and S be weakly commuting mappings and Q and T be weakly commuting mappings of a complete metric space $(X.d)$ into itself satisfying the conditions:

Given $\varepsilon > O$, there exists an $h(\varepsilon) > O$, $h(\varepsilon)$ being a nondecreasing function of ε such that for all x, y in X

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$\varepsilon \leq max\{d(Sx,Ty), d(Px,Sx), d(Qy,Ty),$

$[d(Px,Ty) + d(Qy,Sx)]/2\} < \varepsilon + h$

$=> d(Px,Qy) < \varepsilon$ ....(i)

$Px = Qy$ whenever $Px = Sx$ and $Qy = Ty$. ....(ii)

Let the range of T contain the range of P and the range of S contain the range of Q. If one of the mappings P, Q, S and T is continuous then the mappings P,Q, S and T have a unique common fixed point which is also the unique common fixed point of P and S and of Q and T.

It is of interest to note that in the present extension of the above theorem we have assumed very few additional conditions.

Both of our results extend and unify several important fixed point theorems.

RESULTS

Two selfmappings F and G of a metric space (X,d) are called weakly commuting provided

$d(FGx, GFx) \leq d(Fx,Gx)$

for all x in X.

Theorem 1 Let P_n, Q_n, S_n and T_n be mappings of a metric space (X,d) into itself with at least one common fixed point u_n for each n = 1,2,... . Let P, Q, S and T be selfmsappings of X having a common fixed point u and satisfying the conditions:

Given $\varepsilon > O$, there exists $h(\varepsilon) > O$, $h(\varepsilon)$ being a nondecreasing function of $\in$, such that for all x,y in X

$\varepsilon \leq max \{d(Sx,Ty), d(Px,Sx), d(Qy,Ty), [d(Px,Ty) + d(Sx,Qy)]/2\} < \varepsilon + h$

$=> d(Px,Qy) < \varepsilon$ ....(1)

$Px = Qy$ whenever $Px = Sx$ and $Qy = Ty$. ......(2)

If the sequences of mappings $\{P_n\}$, $\{Q_n\}$, $\{S_n\}$ and $\{T_n\}$ converge uniformly to P, Q, S and T respectively then the sequence $\{u_n\}$ converges to u uniquely.

Proof From (1), for all x, y in X for which $Px, \neq Sx$ and $Qy \neq Ty$, we have

$$d(Px,Qy) < max\ \{d(Sx,Ty),\ d(Px,Sx),\ d(Qy,Ty),$$

$$[d(Px,Ty) + d(Sx,Qy)]/2\} \qquad ......(3)$$

The point u_n is the common fixed point of P_n, Q_n, S_n and T_n fo reach $n = 1,2,\ldots$, so we have

$$P_n u_n = S_n u_n = u_n = Q_n u_n = T_n u_n.$$

Also, $Pu = Su = u = Qu = Tu$.

To establish the theorem, we first show that

$$lim_n\ d(Pu_n, Qu) = 0.$$

If $\lim_n d(Pu_n,Qu) \neq 0$, then there exist a subsequence $\{u_{n_i}\}$ of $\{u_n\}$, a positive integer N and a number $r > 0$ such that

$$n_i \geq N \Rightarrow inf\ \{d(P_{u_{n_i}}, Qu)\} = r.$$

Further, since the sequenes $\{P_n\}$, $\{Q_n\}$, $\{S_n\}$ and $\{T_n\}$ converge uniformly to P, Q, S and T respectively, given $h = h(r) > O$ and for each $u_n \in X$ there exists a positive integer $M = M(h(r))$ such that

$$n \geq M \Rightarrow d(P_n u_n,\ Pu_n) < h/4,\ d(Q_n u_n, Qu_n) < h/4.$$

$$d\,(S_n u_n,\ Su_n) < h/4,\ d(T_n u_n,\ Tu_n) < h/4 \qquad ......(4)$$

Therefore, for $n \geq N$ we obtain

$$d(Pu_n,Qu) < max\{d(Su_n,Tu),\ d(Pu_n,Su_n),\ d(Qu,Tu),\ [d(Pu_n,Tu) + d(Su_n,Qu)]/2\}$$

$$\leq d(S_n u_n, Su_n) + d(S_n u_n, Pu_n) + d(Pu_n, Tu)$$

$$= d(Pu_n, Qu) + d(S_n u_n, Su_n) + d(P_n u_n, Pu_n)$$

$$< d(Pu_n, Qu) + h/2. \qquad \ldots\ldots(5)$$

Now, if we increase N then r and $h(r)$ do not increase, that is, if we increase N then M need not increase. Let us therefore, select $N > M$ and denote the corresponding value of r by r_N, that is,

$$n_i \geq N \Rightarrow \inf\{d(P_{u_{n_i}}, Qu)\} = r_N.$$

This means there exists at least one value m of n_i such that $m \geq N$ and $d(Pu_m, Qu)$ is arbitrarily close to r_N and

$$d(Pu_m, Qu) > r_N. \qquad \ldots\ldots(6)$$

This, in view of (5), implies

$$r_N < \max\{d(Su_m, Tu), d(Pu_m, Su_m), d(Qu, Tu), [d(Pu_m, Tu) + d(Su_m, Qu)]/2\}$$

$$< r_N + h(r_N). \qquad \ldots\ldots(7)$$

However, in view of (1) and (4), inequality (7) yields

$$d(Pu_m, Qu) < r_N.$$

This contradicts (6). Hence $\lim_n d(Pu_n, Qu) = 0$. Then

$$\lim_n \quad d(u_n, u) = \lim_n \quad d(P_n u_n, Qu)$$

$$\leq \lim_n \quad d(P_n u_n, P_n n) + \lim_n \quad d(Pu_n, Qu) = 0$$

Therefore, the sequence $\{u_n\}$ converges to u. Proof of uniqueness of u follows easily.

Thus the theorem is established.

Remark 1 : On taking $S_n = T_n$ and $P_n = Q_n$ for every value of n, in the above theorem, we obtain a generalized version of Theorem 4 of Singh [14] as a special case of our theorem.

Remark 2: In the above theorem, if we take $P_n = Q_n$ and $S_n = T_n$ = the identity mapping, we obtain Theorem 6.12 of Singh [15] as a particular case of our theorem. We also obtain theorem due to Dube and Singh (1972), see Singh [15], as a particular case of our theorem.

Remark 3 : In theorem 1, if we select $P_n = Q_n$ and $S_n = T_n$ = the identity map, then we obtain an extended form of the Theorem 4.6.8 of [4, p. 151] which is due to Bose and Mukherjee [1].

Theorem 2 : Let $\{P_i\}$ and $\{Q_j\}$, $i,j = 0,1,2, \ldots,$ be sequences of selfmappings and let S and T be selfmappings of a complete metric space (X,d) satisfying the conditions:

Given $\varepsilon > 0$, there exists $h(\varepsilon) > 0$, $h(\varepsilon)$ being a nondecreasing function of ε, such that for all x, y in X

$$\epsilon \le max\{d(Sx,Ty), d(P_i x,Sx), d(Q_j y,Ty), [d(P_i x,T_y) + d(Q_j y,Sx)]/2\} < \epsilon + h$$

$$\Rightarrow d(P_i x, Q_j y) < \epsilon \qquad \ldots\ldots(1)$$

$P_i x = Q_j y$ whenever $P_i x = Sx$ and $Q_j y = Ty$. ......(2)

Let the range of one of the mappings $\{P_i\}$, say Po, be contained in the range of T and the range of one of the mappings $\{Q_j\}$, say Q_0, be contained in the range of S. Also let P_0 and S, and Q_0 and T be weakly comuting. If one of the mappings P_0, Q_0, S and T be continuous then the mappings $\{P_i\}$, $\{Q_j\}$, S and T have a unique common fixed point which is also the unique common fixed point of $\{P_i\}$ and S and of $\{Q_j\}$ and T for every value of i and j.

Proof Let us first consider the mappings P_o, Q_o, S *and* T. These four mappings satisfy all the conditions of Theorem A. Therefore, by virtue of Theorem A, the mappings P_o, Q_o, S and T have a unique common fixed point, say z. Further, z is the unique common fixed point of P_o and S and of Q_o and T. Thus

$$P_o z = Q_o z = z = Sz = Tz.$$

Let us now consider a mapping P_i for some arbitrarily chosen value of i. Then $z = P_i z$. Otherwise, from (1) we get

$$d(P_i z,\ z) = d(P_i z, Q_o Z)$$

$$< max\{d(Sz,Tz),\ d(P_i z,Sz),\ d(Q_o z,Tz),$$

$$[d(P_i z,Tz) + d(Q_o z,Sz)]/2\}$$

$$= d(z,P_i z),$$

a contradiction. Similarly it can be shown that $z = Q_j z$ for any value of j. If possible, suppose u is a second common fixed point of P_i and S. Then from (1) we obtain

$$d(u,z) = d(P_i u,Q_0 z)$$

$$< max\{d(Su,Tz),\ d(P_i u,Su),\ d(Q_o z,Tz),\ [d(P_i u,Tz) + d(Q_o z,Su)]/2\}$$

$$= d(u,z),$$

a contradiction. Hence z is the unique common fixed point of P_i and S for each value of i. Similarly z is the unique common fixed pont of Q_j and T for every value of j.

This establishes the theorem.

Remark 4 : In Theorem 2, if we let $P_i = Q_i$ for every value of i and $S = T =$ the identity mapping, we obtain a generalized version of a theorem due to Bose and Mukherjee [1].

Remark 5 : Letting $P_i = Q_i$ for each i and $S = T$ in Theorem 2, an extended form of Theorem 1 of Tewari and Singh [16] is obtained.

Remark 6 : By an obvious modification in the proof of Theorem 2, that is, by using the properties of compatibility in place of those of weak commutativity at relevant places, the theorem pertains to compatible maps and we obtain a multitude of fixed point theorems as special cases of our theorem. For example, if we take $P_i = P$ for each i, we can obtain the theorem due to Rhoades, Park and Moon [13] under relaxed conditions as a special case of our theorem. The theorem of Rhoades, Park and Moon assumes stronger condition on compatibility and the ranges of the mappings. The theorems due to Fisher [2], Hardy and Rogers [3], Jungck [5], Kannan [6], Meir and Keeler [7], Pant [8, p. 78], [9], [10], Park and Bae

[11], Park and Rhoades [12], Singh [14], Tewari and Singh [16], [17] and Theorem 4.5.8 of [4] can also be obtained as special cases.

Remark 7 : A weaker form of Theorem 2, under the stronger condition that each P_i weakly commutes with S and each Q_j weakly comutes with T, can be obtained as a special case of Theorem 2.15 of [8, p. 89).

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 175-182

A FIXED POINT THEOREM FOR SELF MAPPINGS ON MENGER SPACES

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(Received 02.12.1994)

ABSTRACT

In this paper a fixed point theorem in the context of Menger Spaces is proved. The theorem is supported by an example. A fixed point theorem in complete metric spaces also follows.

AMS Subject Classification (1990) Primary 54H25.

Keywords : Menger Space, fixed point, t-norm.

INTRODUCTION

A Menger Space is defined as a triplet $(M,F,t,)$ where M is a set, $F{:}MxM \to \Delta^+$ (set of distribution functions) satisfies the following

(a) $F_{xy}(0) = 0$, for all $x,y \,\varepsilon\, M$
(b) $F_{xy}(s) = 1$, for all $s>0$ holds if and only if $x=y$
(c) $F_{xy} = F_{yx}$, for all $x,y \,\varepsilon\, M$
and (d) $F_{xy}(s_1 + s_2) \geq t(F_{xz}(s_1), F_{zy}(s_2))$, for all $x,y,z \,\varepsilon\, M$ and $s_1, s_2 \geq 0$

where t is a t-norm, that is a function $t{:}[0,1]x\,[0,1] \to [0,1]$ which satisfies
(i) $t(a,1) = a, t(0,0) = 0$
(ii) $t(a,b) = t(b,a)$
(iii) $t(c,d) \geq t(a,b)$, $c \geq a$, $d \geq b$
and(iv) $t(a,t(b,c)) = t(t(a,b),c)$ for all $a,b,c,d \,\varepsilon\, [0,1]$

Examples of t-norm are $t(a,b) = min(a,b), t(a,b) = max\{a+b-1,0\}$etc.

A Sequence $\{p_n\}$ is said to converge to p if given $\varepsilon>0$, $\lambda>0$ there exists a positive integer $N_{\varepsilon,\lambda}$ such that $F_{pp_n}(\varepsilon) > 1-\lambda$ whenever $n \geq N_{\varepsilon,\lambda}$.

A Sequence $\{p_n\}$ is a Cauchy sequence if given $\varepsilon>0$, $\lambda>0$ there exists

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a positive integer $N_{\varepsilon,\lambda}$ such that $F_{p_m p_n}(\varepsilon) > 1-\lambda$ whenever $m,n > N_{\varepsilon,\lambda}$.

A Menger Space is complete if every Cauchy sequence is convergent.

MAIN RESULTS

In this section we obtain a fixed point theorem for a self-mapping on a complete Menger space.

Lemma-1. $\{p_n\}$ is a sequence in a Menger Space $(M, F, t \equiv min)$ and $\{s_n\}$ is a sequence of positive real numbers such that $\sum_{i=0}^{\infty} s_i < \infty$ If $\pi_{i=0}^{\infty} F_{p_i p_i}(s_i) > 0$ then $\{P_n\}$ is a Cauchy sequence.

Proof. None of $F_{p_i p_{i+1}}(s_i) = 0$. So let, $F_{p_i p_{i+1}}(s_i) = 1 - \lambda_i$ where $0 \le \lambda_i < 1$, i = 1, 2..... Then, $\pi_{i=1}^{\infty} (1-\lambda_i) > 0$ this implies ([3]) $\sum_{i=1}^{\infty} \lambda_i < \infty$. Given $\varepsilon > 0$ and $\lambda > 0$ we can find a positive integer $N_{\varepsilon,\lambda}$ such that

$$\sum_{i=N_{\varepsilon,\lambda}}^{\infty} s_i < \varepsilon \text{ and } \sum_{i=N_{\varepsilon,\lambda}}^{\infty} \lambda_i < \lambda$$

Let $m, n > N_{\varepsilon,\lambda}$. Further let $n > m$.

$$F_{p_m, p_n}(\varepsilon) \ge F_{p_m p_n}(s_m + \ldots + s_{n-1})$$

$$\ge t\left(F_{p_m p_{m+1}}(s_m), F_{p_{m+1} p_n}(s_{m+1} + \ldots + s_{n-1})\right)$$

$$\ge t\left(F_{p_m p_{m+1}}(s_m), t\left(F_{p_{m+1} p_{m+2}}(s_{m+1}), t\left(\begin{array}{l} F_{p_{m+2}, p_{m+3}}(s_{m+2}) \ldots, \\ t\left(F_{p_{n-2}, p_{n-1}}(s_{n-2})\, F_{p_{n-1} p_n}(s_{n-1})\right)\end{array}\right)\right)\right)$$

By virtue of properties of t-norm it follows that

$$F_{p_m p_n}(\varepsilon) \ge min\left\{F_{p_m p_{m+1}}(s_m), F_{p_{m+1} p_{m+2}}(s_{m+1}) \ldots, F_{p_{n-1}, p_n}(s_{n-1})\right\}$$

$$\ge min\{1-\lambda_m, 1-\lambda_{m+1} \ldots, 1-\lambda_{n-1}\}$$

or $F_{p_m p_{m+1}}(\varepsilon) > 1 - \lambda$ whenever $m, n > N_{\varepsilon, \lambda}$. Thus $\{p_n\}$ is a Cauchy sequence.

Theorem-1. *Let* $(M, F, t \equiv min)$ be a complete menger space. T:M→M satisfies the following

$$\text{a)} \qquad F_{T_x, T_y}(s_1 + s_2) \geq F_{x, T_x}\left(\frac{s_1}{\alpha}\right) + F_{T_y, T_y^2}\left(\frac{S_2}{\beta}\right) - 1$$

where $0 < \alpha < 1$ and $\beta > 0$ are constants and $s_1, s_2 \geq 0$
and b) there exists $\{x_i) \subset M$ and a sequence of real numbers $\{s_i\}$ with $\sum_{i=1}^{\infty} s_i < 1$ such that

$F_{x_1, T_{x_1}}(s_1) > 1 - s_1$

$F_{T_{x_2}, T^2{}_{x_2}}(s_2) > 1 - s_2$

and in general $F_{T_{x_i}^{i-1}, T_{x_i}^{i}}(s_i) > 1 - s_i$, $i = 1, 2\ldots$
Then, T has a unique fixed point.

Proof. $F_{T_{x_i}^i, T_{x_{i+1}}^i}(s_i + \beta s_{i+1}) \geq F_{T_{x_i}^{i-1}, T_{x_i}^i}\left(\frac{s_i}{\alpha}\right) + F_{T_{x_{i+1}}^i, T_{x_{i+1}}^{i+1}}\left(\frac{\beta s_{i+1}}{\beta}\right) - 1$

Since, $0 < \alpha < 1, F_{T_{x_i}^i, T_{x_{i+1}}^i}(s_i + \beta s_{i+1}) \geq F_{T_{x_i}^{i-1}, T_{x_i}^i}(s_i) + F_{T_{x_{i+1}}^i, T_{x_{i+1}}^{i+1}}(s_{i+1}) - 1$

$> 1 - s_i + 1 - s_{i+1} - 1$

$= 1 - s_i - s_{i+1}$

Thus we have,

$F_{x_1, T_{x_1}}(s_1) > 1 - s_1$

$F_{T_{x_1}, T_{x_2}}(s_1 + \beta s_2) > 1 - (s_1 + s_2)$

$F_{T_{x_2},T^2_{x_2}}(s_2) > 1 - s_2,$

$F_{T^2_{x_2},T^2_{x_3}}(s_2 + \beta s_3) > 1 - (s_2 + s_3),$

$F_{T^2_{x_3},T^3_{x_3}}(s_3) > 1 - s_3$

$F_{T^3_{x_3},T^3_{x_4}}(s_3 + \beta s_4) > 1 - (s_3 + s_4$ ....etc.

Also $0 < s_1, (s_1+s_2), s_2, (s_2+s_3)$........< 1 because $\sum_{i=1}^{\infty} s_i <$

Now the series, $s_1 + (s_1+s_2) + s_2 + (s_2+s_3) + (s_3+s_4) +$....to
$= 2s_1 + 3s_2 + 3s_3 +$..................to is convergent

So the infinite product,

$(1-s_1)(1-s_1-s_2)(1-s_2)(1-s_2-s_3)$.....to $\infty > 0$ ([3])

Hence, $F_{x_1,T_{x_1}}(s_1)\, F_{T_{x_1},T_{x_2}}(s_1 + \beta s_2).F_{T_{x_2},T^2_{x_2}}(s_2)$
$.F_{T^2_{x_2},T^2_{x_3}}(s_2 + \beta s_3)$....to $\infty > 0$

Also the series,

$s_1 + (s_1 + \beta s_2) + s_2 + (s_2 + \beta s_3) + s_3 + (s_3 + \beta s_4) + s_4 +$.... to ∞.

$= 2s_1 + (2 + \beta)\, s_2 + (2 + \beta)\, s_3 +$ ......to ∞

is convergent, since $\sum_{i=1}^{\infty} s_i <$

Therefore by Lemma-1 the sequence

$\{ x_1, T_{x_1}, T_{x_2}, T^2_{x_2}, T^2_{x_3}, T^3_{x_3}, T^3_{x_4}, T^4_{x_4} \ldots T^{i-1}_{x_i}, T^i_{x_i}, \ldots \}$

is convergent. Let the sequence converge to x.

Let ε > 0 be arbitrary.

We choose α_1 such that $\alpha < \alpha_1 < 1$,

then $F_{xT_x}(\varepsilon) \geq t(F_{xT^{i-1}_{x_i}}((1-\alpha_1)\varepsilon)), F_{T^{i-1}_{x_i},T_x}(\alpha_1\varepsilon))$

$= t(F_{xT^{i-1}_{x_i}}((1-\alpha_1)\varepsilon), F_{T^{i-1}_{x_i},T_x}(\alpha_2+(\alpha_1-\alpha_2)\varepsilon)$

(where we choose α_2 such that $\alpha < \alpha_2 < \alpha_2$)

$\geq t(F_{x,T^{i-1}_{x_i}}((1-\alpha_1)\varepsilon), F_{x,T_x}(\frac{\alpha_2}{\alpha}\varepsilon) + F_{T^{i-1}_{x_i},T^{i}_{x_i}}(\frac{\alpha_1-\alpha_2}{\beta}\varepsilon))-1)$

Let n be an arbitrary positive integer. Since the sequence $\{x_1, T_{x_1}, T_{x_2}, T^2_{x_2}, T^2_{x_3}, T^3_{x_3}\ldots\}$

converges to the point x we can choose i in such a way that

$F_{T^{i-1}_{x_i},T^{i}_{x_1}}(\frac{\alpha_1-\alpha_2}{\beta}\varepsilon) > 1-\frac{1}{n}$

and $F_{xT^{i-1}_{x_i}}((1-\alpha_1)\varepsilon) > 1-\frac{1}{n}$,

Therefore $F_{xT_x}(\varepsilon) \geq t(1-\frac{1}{n}, F_{xT_x}(\frac{a_2}{\alpha}\varepsilon)+1-\frac{1}{n}-1)$

$\geq t(1-\frac{1}{n}, F_{xT_x}(\frac{a_2}{\alpha}\varepsilon)-\frac{1}{n}$

$\geq t(F_{xT_x}(\frac{a_2}{\alpha}\varepsilon)-\frac{1}{n}, F_{xT_x}(\frac{a_2}{\alpha}\varepsilon)-\frac{1}{n})$

$F_{xT_x}(\varepsilon) \geq F_{xT_x}(\frac{a_2}{\alpha}\varepsilon)-\frac{1}{n}$

Since n is arbitrary

$F_{xT_x}(\varepsilon) \geq F_{xT_x}(\frac{a_2}{\alpha}\varepsilon)$

Since $\frac{a_2}{\alpha} >$ we have $F_{xT_x}(\varepsilon) = 1$ for all $\varepsilon > 0$.
That is $x = T_x$ is a fixed point. If possible, let $y = T_y$ be another fixed point of T.

Then from a) for $s > 0$

$$F_{T_x,T_y}(s) \geq F_{x,T_x}\left(\frac{s}{2\alpha}\right) + F_{T_y,T_y^2}\left(\frac{s}{2\beta}\right) - 1$$

or $$F_{x,y}(s) \geq F_{x,x}\left(\frac{s}{2\alpha}\right) + F_{y,y}\left(\frac{s}{2\beta}\right) - 1$$

or $F_{x,y}(s) \geq 1$ for $s > 0$.

Hence $x = y$
Therefore the fixed point is unique.

Cor-1. If (X,d) is a complete metric space and $T : \rightarrow X$ satisfies
1. $d(T_x, T_y) < \alpha d(x, T_x) + \beta d(T_y, T^2_y)$, where $0 < \alpha < 1$ and $\beta > 0$
11. T is onto
111. $\inf d(x, T_x) = 0$
Then T has unique fixed point

The proof of the Corollary will follow from the fact that a complete metric space can also be treated as a complete menger space if we put $F_{pq}(t) = H(t - d(p, q))$ [4], [6]. The inequality (1) in corollary implies the inequality (a) in the Theorem and conditions (11) and (111) in the corollary imply condition (b) of the Theorem. The proof then follows by the application of Theorem 1.

It may noted that fixed points on Menger space have been discussed in many works. We have noted some of them in references [1], [2] [5], [6].

Examples

We set an example to show the applicability of Theorem-1.
Let G any distribution function with $G(0)$ abd $0 < G(x) < 1$ for all $x > 0$. We consider the triplet (M,F,t) where $M = [0, \infty]$ t = min and F is defined as

$F_{pq} = H$, if $p = q$

where $H(x) = 1, x . 0$
$= 0, x < 0$
and $F_{pq}(s) = G(s/ \ |p - q|)$
It is easily seen that (M,F,t) is a complete Menger space.

Let $T : X \to X$ be defined as $T_x = x/4$. Then

$$|T_x - T_y| = \left|\frac{x}{4} - \frac{y}{4}\right| = \frac{1}{4}\ |x - y| \le \frac{1}{4}(|x| + |y|)$$

or, $|T_x - T_y| \le \frac{1}{4}.\ 2\ max\ \{|x|\ ,\ |y|\}.$

Therefore either Case I : $|T_x - T_y| \le \frac{1}{2}\ |x|$

or Case II : $|T_x - T_y| \le \frac{1}{2}\ |y|$

Case I : $|T_x - T_y| \le \frac{1}{2}\ |x| = \frac{1}{2}\frac{4}{3}\left|\frac{3}{4}x\right|$

$$\frac{2}{3}\left|x - \frac{1}{4}x\right| = \frac{2}{3}|x - T_x|$$

$$G\left(\frac{s}{|T_x - T_y|}\right) \ge G\left(\frac{s}{\frac{2}{3}|x - T_x|}\right),\ s > 0$$

$${}^F T_x, T_y(s_1 + s_2) \ge F_x, T_x\left(\frac{s_1}{2/3}\right) + {}^F T_y,\ T_y^2\left(\frac{s_2}{8/3}\right)$$

Case II : $|T_x T_y| \le \frac{1}{2}|y| = \frac{1}{2}\frac{16}{3}\left|\frac{3}{16}y\right|$

$$= \frac{8}{3}|y/4 - y/16| = \frac{8}{3}|T_y - T_y^2|$$

so $G(\frac{s}{|T_x - T_y|}) \ge G(\frac{s}{8/3|T_y - T_y^2|})s > 0$

or $F_{T_x T_y}(s) \ge F_{T_y T_y^2}(\frac{s}{8/3})$

Both cases imply inequality

$$F_{T_x, T_y}(s_1 + s_2) \ge F_x, T_x\left(\frac{s_1}{2/3}\right) + F_{T_y},\ {}_{T_y^2}\left(\frac{s_2}{8/3}\right)$$

$$\geq F_{x, T_x}\left(\frac{s_1}{2/3}\right) + F_{T_y, T_y^2}\left(\frac{s_2}{8/3}\right) - 1$$

where $s_1, s_2 > 0$.

Let $\{s_1\}$ be a sequence such that $\sum_{i=1}^{\infty} s_i <$. we choose $\{x_i\}$ such that

$$G(s_i / \left| x_i(\frac{1}{4^i} - \frac{1}{4^{i-1}}) \right|) > 1 - s_i,\ i = 1,2$$

so that $F_{T^{i-1}_{x_i} T^i_{x_i}}(s_i) > 1 - s_i.\ i = 1,2...$

Hence b) of Theorem-1 is satisfied with the choice of sequence $\{x_i\}$ and $\{s_i\}$

Thus by application of Theorem-I it follows that T has a unique fixed point. It may be seen that `$O$` is the unique fixed point.

ACKNOWLEDGEMENT

I am grateful to Prof. J.N. Das, Deptt. of Applied Mathematics, Calcutta University, for his guidance and advice during the preparation of the paper.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 183-186

FIXED POINTS IN FUZZY METRIC SPACES

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(Received 21-12-1994)

ABSTRACT

In this note we establish Banach type fixed point theorems on fuzzy metric spaces (FM-spaces). Our results generalize several known fixed point theorems on FM-spaces and extend certain results of Hicks-Rhoades and others.

Mathematics Subject Classifications (1991) : 54H25, 54A40, 54E99.

Keyword and Phrases : Fuzzy metric space, fixed point, orbitally upper semicontinuous.

INTRODUCTION

Following Kramosil-Micha'lek's [5] approach to fuzzy metric spaces (FM-spaces), Grabiec [2] has formulated fixed point theorems for contractive maps on FM-spaces. Subsequently, Fang [1], Mishra et al. [6] and Singh et al. [7] have investigated existence of solutions of abstract fixed point equations on FM-spaces. The intent of this note is to establish fixed point theorems for Banach type maps on FM-spaces. Our results extend and fuzzify fixed point theorems of Hicks-Rhoades [3], Hicks [4] and others.

PRELIMINARIES

We shall generally follow the notations and definitions of Grabiec [2], Hicks-Rhoades [3], and Mishra et al. [6].

A fuzzy metric space is an ordered triplet $(X, M, *)$ consisting of a nonempety set X, a fuzzy set M in $X^2 \times (0, \infty)$ and $*$, a continuous t-norm [2] (see, also [5], [6]). The functions $M(x, y, .) : [0, \infty) \to [0, 1]$ are left continuous and are assumed to satisfy the following conditions :

(FM-1) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$

(FM-2) $M(x, y, 0) = 0$

(FM-3) $M(x, y, t) = M(y, x, t)$

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(*FM-4*) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$, *for all* $x, y, z \in X$ and $t, s > o$.

Grabiec [2] has shown that $M(x, Y, *)$ is nondecreasing for all $x, y \in X$. For details of topological preliminaries, refer to [2] and [5].

A sequence $\{x_n\}$ in X is convergent to $x \in X$ if $\lim M(x_n, x, t) = 1$ for each $t > o$, and $\{x_n\}$ is a Cauchy sequence if $\lim M(x_{n+p}, x_n, t) = 1$ for each $t > 0$, and $p > 0$. An FM-space, in which every Cauchy sequence is convergent, is called complete.

In all that follows, X stands for an FM-space $(X, M, *)$ with the following property :

(*FM-5*) $\lim_{t \to \infty} M(x, y, t) = 1$ *for all* $x, y \in X$.

FIXED POINT THEOREMS

Defination 1. Let X be an FM-Space and $T : X \to X$. Orbit of T at x, denoted by $O(x, \infty)$ is defined as $O(x, \infty) = \{x, Tx, T^2x.....\}$.

Defination 2. Let X be an FM-space. A function G from X to $[0, 1]$ is *T-orbitally upper semicontinuous at* $z \in X$ if $\{x_n\}$ is a sequence in $O(x, \infty)$ and $\{x_n\}$ converges to z implies $G(z) \geq \lim\sup G(x_n)$.

The following result is essentially due to Grabiec [2].

Theorem 1. Let X be an FM- space. If $T: X \to X$ satisfies

(1.1) $M(Tx, Ty, kt) \geq M(x, y, t)$;

for all $x, y \in X$, *and* $o < k < l$, *then T has a unique fixed point.*

The above result is known as *fuzzy Banach contraction principle.*

Theorem 2. *Let X be an FM-space and* $T : X \to X$. *If X is T-orbitally complete and for an* $x \in X$,

(2 .1) $M(Ty, T^2 y, kt) \geq M(y, Ty, t)$;

$y \in O(x, \infty)$, $k \in (0, 1)$, *then for an* $x_o \in X$, *there exists an orbit* $O(x_o)$ *converging to a point* $z \in X$. *Further :*

(*a*) *z is a fixed point of T if T is orbitally continuous.*

(b) z is a fixed point of T if and only if $G(x) = M(x, Tx, t)$ is T-orbitally upper semicontinuous at z.

Proof. Pick $x_0 \in X$. We construct a sequence $\{x_n\}$ in X such that $x_1 = Tx_0$, $x_2 = Tx_1, \ldots, x_n = Tx_{n-1} = T^n x_0$
In view of (2.1),

$$\begin{aligned} M(x_n, x_{n+1}, t) &= M(Tx_{n-1}, T^2x_{n-1}, t) \\ &\geq M(x_{n-1}, Tx_{n-1}, t/k) \\ &\geq \ldots \geq M(x_0, x_1, t/k^n). \end{aligned}$$

This shows that $\{x_n\}$ is a cauchy sequence in X, hence convergent. Call the limit z.

(a) Since
$M(Tz, z, t) \geq M(Tz, Tx_n, t/2) * M(x_{n+1}, z, t/2)$

and T is continuous, making $n \to \infty$ yields $M(Tz, z, t) \geq 1*1 = 1$

Thus z is a fixed point of T.

(b) Let $z = Tz$ i.e $M(z, Tz, t) = 1$ Then $G(z) = M(z, Tz, t) = 1 \geq \lim\sup G(x_n)$, implying $G(x)$ is T-orbitally upper upper semicontinuous at z.

Conversely, if $G(x)$ is T-orbitally upper semicontinuous at z, then $G(x_n) = M(x_n, Tx_n, t) \leq M(z, Tz, t)$, yielding $M(z, Tz, t) = 1$, i.e., $z = Tz$.

Following theorem shows that Theorem 2 is more general than Theorem 1.

Theorem 3. *Let X be an FM-space and $T: X \to X$. Then the condition (1.1) implies the condition (2.1)*

Proof. Taking $y = Tx$ in (1.1), we get the condition (2.1).

Here we remark that our result is superior to several results on the existence of fixed points in FM-spaces. But, first, consider the following for $T: X \to X$. For all $x, y \in X$,

$$\text{(3.1)} \quad \begin{aligned} M(Tx, Ty, kt) \geq\ & M(x, y, t) * M(x, Tx, t) * M(y, Ty, t) * \\ & M(x, Ty, \alpha t) * M(y, Tx, (2-\alpha)t), \end{aligned}$$

where $k \in (0, 1)$, $t > 0$ and $\alpha \in (0, 2)$.

Recently Mishra et al. [6] and Singh et al. [7] studied the condition (3.1) and obtained some interesting results on FM-spaces. Following comparing theorem shows that our result also generalize several fixed point theorems for above contractive condition.

Theorem 4. Let X be an FM-space and $T : X \to X$. Then the condition (3.1) implies the condition (2.1).

Proof. Taking $y=Tx$ in (3.1). We get for $\alpha = 1+q$, $q \in (0, 1)$

$$\begin{aligned} M(Tx,\ T^2x,\ kt) &\geq M(x,\ Tx,\ t) * M(Tx,\ T^2x,\ t) * M(x, T^2x, \\ &\qquad (1+q)\,t) * M(Tx,\ Tx,\ (1-q)\,t) \\ &\geq M(x,\ Tx,\ t) * M(x,\ Tx,\ qt). \end{aligned}$$

Since the norm * is continuous and $M(x, y, .)$ is left continuous, making $q \to 1$ gives $M(Tx, T^2x, kt) \geq M(x, Tx, t)$. This completes the proof.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp.187-190

ON SOME PROPERTIES OF DISTANCE SETS

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(Received 21-12-94)

ABSTRACT

Steinhaus proved that the set of differences A-B contains an interval whenever A and B are linear sets of positive Lebesgue measure. S.Piccard proved that the same conclusion holds when A and B have the property of Baire. In the present note we strengthen the results of Steinhaus as well as Piccard.

Mathematics subject classification (1991): 28A 05

Key words and Phrases : Distance set, Property of Baire, Residual set.

INTRODUCTION

Let A and B be subsets of the real line R. Then the distance set of A and B is defined to be the set $D(A,B) = \{|x-y|: x \in A, y \in B\}$. The investigation of the set of distances between the points of a measurable set was started by H. Steinhaus. In 1920, Steinhaus [7] proved that if $A, B \subset R$ be sets with positive Lebesgue measure then $D(A,B)$ contains an interval. A linear set is said to have the "Property of Baire" if it can be expressed as the symmetric difference of an open set and a set of first category. A set of the second category which is complement to a set of the first category is called a residual set [4]. Therefore, there are two kinds of sets of the second category, those which are complement to a set of the first category and those for which this is not the case. It is also known that the intersection of a sequence of residual sets is also residual. In 1939, S. Piccard [5] proved the result same as that of Steinhaus when A and B were two linear sets of second category having the property of Baire. Many papers are devoted to the study of the distance set $D(A)$ for various A. The corresponding problem for sets in n-dimensional space was considered by Kestelman [3] in 1947, and he proved a number of elegant theorems about distance sets in n-dimensional space and Steinhaus theorem came out as a particular case of one of his results. The question has been asked if there is

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a set of measure zero for which the distance set contains an interval with origin as a left hand end point. In 1917, Steinhaus [6] proved that the Contor set C is an example of a set of measure zero whose distance set contains an interval with origin as a left hand end point. Bose Majumdar [2] made an exhausted study of $D(C)$. Boas [1] has proved that for almost all d in $[0,1]$ there is an uncountable (of the power of the continuum) set of pairs $(x,y) \in C \times C$ such that $|x-y| = d$. In connection with this, the question arises what is the situation for the sets of positive measure.

Steinhaus [7] proved the following interesting result: *Let A and B be linear sets with positive measure and C be a set of real numbers everywhere dense; then there exist two points 'a' and 'b' belonging to A and B respectively such that the number $|a-b|$ belongs to C.* As a consequence of this result he proved that $D\ (A,B)$ for sets of positive measure A and B, contains an interval.

In this paper we have attempted to generalise this result. For some d in a countable dense subset of R, the equality $|x-y| = d$ holds for an uncountable (of the power of the continuum) number of pairs $(x,y) \in A \times B$. We obtain the same result as above when A and B are taken as sets of second category having the property of Baire.

Theorem 1. *Let A and B be two linear sets of positive measure and C be a countably dense subset of R. Then for some $d \in C$, the set $E = \{x \in A : \exists\, y \in B$ with $|x-y| = d\}$ has positive measure.*

Proof. If possible let E be of measure zero. Then for each $d \in C$, the set $\{x \in A:\ \exists y \in B$ such that $|x-y| = d\}$ has measure zero. Therefore the set $F = \{x \in A:$ $\exists y \in B$ such that $|x-y| = d,\ d \in C\}$ also has measure zero.

Let $H = A \setminus F$. Then $\lambda(H) = \lambda(A) > 0$, where λ denotes the Lebesgue measure. Then according to Steinhaus [7] the set $B \setminus H$ contains an interval. Hence $C \cap (B \setminus H) \neq \Phi$. Then for some $y \in B,\ c \in C$ and $x \in H \subseteq A$ we have $y = x + c$. Thus $x \in F$-a contradiction. Hence the result.

Note. For some d in a countably dense subset C of R, the equality $|x-y| = d$ holds for an uncountable (of the power of continuum) number of pairs $(x,y) \in A \times B$.

Steinhaus [7] proved that $D(A,B)$ contains an interval where A and B are two linear sets of positive measure. Let us define the set $D_1(A,B)$ consisting of all those points d which can be expressed into the form $d=|x-y|$ for infinite number of pairs of points $x\in A,\ y\in B$. Obviously, $D_1(A,B)\subset D(A,B)$.

We have established the following result which is an easy consequence of Steinhaus's result :-

Theorem 2. If A and B are two linear sets of positive measure, then $D_1 \equiv D_1(A,B)$ contains at least one interval.

Proof. Suppose not, then D_1 does not contain an interval. So $R\backslash D_1$ is a dense subset of R. Then, we may choose C to be countably dense subset of $R\backslash D_1$. By the note after theorem 1 there exists $d\in C$ such that d is the distance between infinite number of points of A and B. Hence $d\in D_1$--*a* contradiction.

Therefore, D_1 contains at least one whole interval.

Theorem 3. *Let A and B be two linear sets of second category having the property of Baire. If C is a countable dense subset of R, then for some* $d\in C$, *the set* $\{x\in A: x\pm d\in B\}$ *is of second category.*

Proof. If not, then for every $d\in C$, the set $F_d=\{x\in A: x\pm d\in B\}$ is of first category.

Then the set $\bigcup_{d\in C} F_d$ is also of first category in R.

Therfore $E = A - \bigcup_{d\in C} F_d$ is a residual set in R.

Then according to Piccard [5], $B\backslash E$ contains an interval. Hence, $C\cap(B\backslash E)\neq\Phi$. Then for some $d\in C,\ b\in B$ and $a\in E\subseteq A$ we have $b=a+d$ and hence $a\in F_d$--*a* contradiction.

Hence the result.

Theorem 4. *Let C be a countable dense subset of R and I be a set of first category. Then there exists a residual set P such that,* $\forall x\in P,\ x+c\notin I$ *for any* $c\in C$.

Proof. Let us enumerate C as $e_1, e_2, \ldots\ldots, e_n, \ldots$. Let $R_n=\{r\in R: e_n+r\notin I\}$ $(n=1,2,\ldots)$. Then $R_n=[R-I]\ (-e_n)$, which is a residual set for every n. Let

$P = \bigcap_{n=1}^{\infty} R_n$ Then P is a residual set and $\forall x \in P,\ x+e_n \notin I$ for any n.

Hence the proof.

Theorem 5. *If A and B are two residual linear sets in some open interval, then the set of all points d for which* $\{x \in A : x+d \in B\}$ *is of second category, contains an interval.*

Proof. Let E be the set of all those points d for which the set $\{x \in A : x+d \in B\}$ is of second category.

We first show that E hits every countable dense subset C of R.

If not, then for each $d \in C$, the set $\{x \in A : x+d \in B\}$ is of first category. Then set $F=\{x \in A : x+d \in B,\ d \in C\}$ is also of first category. Then $G=R \backslash F$ is a residual set having the property of Baire. So according to Piccard [5], $B \backslash G$ contains an interval. Then $(B \backslash G) \cap C \neq \Phi$, which gives rise to a contradiction.

Now, if E contains no interval, then there exists a countable dense subset C of R such that $E \cap C = \Phi$-- which is again a contradiction.

Hence the theorem.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 191-206

NONLINEAR HYBRID CONTRACTIONS

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(Received 30-03-1993)

ABSTRACT

Standard fixed point theorems state conditions under which there exist solutions to $Tx = x$ for the single-valued self-map T of a space X or to $y \in Py$ for a multivalued map P on X with values in its power set. In this paper, inter alia, we present a brief historical account of coincidence/fixed point theorems for contracting single- and multi-valued maps.

Mathematics Subject Classifications (1991) : 54H25, 47HIO.

Keywords and Phrases Coincidence/fixed point, multivalued map, weakly commuting maps, compatible maps.

INTRODUCTION

The well-known Brouwer fixed point theorem (*viz. a continuous map on a closed unit ball of R^n has a fixed point*) has maintained its magnificient presence in mathematics for the last 80 years, excluding, of course, a gestation of 2-3 years. This "fixed point"magnet may be called the queen of fixed point theory and, perhaps, of macro-analysis and its applications. It has many beautiful children, grand and grand-grand children such as, with the risk of missing many champion theorems, Birkhoff-Kellog (1922). Lefschetz (1926). Schauder (1927). Tychonoff (1935). Kakutani (1941), Hukuhara (1950), Glicksberg (1952), Darbo (1955), Browder (1959, 67), Jones (1963), Sadovskii (1967), Ky Fan (1961, 66, 69, 84). Reih (1972, 78, 79), Lin (1979), V.M. Sehgal & S.P. Singh (1985) and Bridges et al. (1992, cf. [5]). One may refer [41]) and [52] for the recent development of these theorems. However, beauty (of these results) has to pay the price of

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*A major part of this article presented at the International Conference on Modern Analysis and Applications, I.I.T., New Delhi (11-14 December 1992).

cosmetics in kind of "compactness in some sense." " - - - the spaces which have the fixed point property for continuous copact set-valued mappings constitute a fairly small subclass of those which has the fixed point proerty for continuous single-valued mappings; (see Nadler [33, p. 476])." This, perhaps, inspires the development of fixed point theory for contracting maps. In this paper, we are concerned about the study of fixed points of contracting multivalued maps "majorized" by single-valued maps, which started about a decade ago. A *prima facie* love for undertaking such a study appears to be H.W. Corley's formulation [7, Th. 1] of an abstract maximization problem in terms of a (hybrid) stationary point theorem.

PRELIMINARIES

We will use the following notation where (X, d) is a metric space :

$$CL(X) = \{A \subset X: A \text{ is nonempty and closed}\},$$

$$CB(X) = \{A \subset X: A \text{ is nonempty, closed and bounded}\},$$

For nonempty subsets A, B of X and $s > 0$,

$$N(s, A) = \{x \in X : d(x, a) < s \text{ for some } a \in A\},$$

$$E_{A,B} = \{s>0 : A \subset N(s,B),\ B \subset N(s,A))\},$$

$$H(A,B) = \begin{cases} \inf E_{A,B} & \text{if } E_{A,B} \neq \phi \\ +\infty & \text{if } E_{A,B} = \phi, \end{cases}$$

and for any $x \in A$, $d(x, A)$ denotes the distance between x and A. H is called the *Hausdorff* (resp. *generalized Hausdorff*) distance function for $CB(X)$ (resp. $CL(X)$) induced by d. In all that follows *id* stands for the identity map on X.

Definition 1 [34], [48]

Maps $f: X \to X$ and $T: X \to CL(X)$ are weakly commuting at $z \in X$ if $H(fTz, Tfz) \leq d(fz, Tz)$. f amd T commute weakly on X if they commute weakly at every point of X.

For an equivalent formulation for two single-valued maps on a metric space, see [44].

Definition 2 [24]

Maps $f : X \to X$ and $T : X \to CB(X)$ are compatible iff $fTx \in CB(X)$ for all $x \in X$ and $H(Tfx_n, fTx_n) \to 0$ whenever x_n is a sequence in X such that $Tx_n \to M \in CB(X)$ and $fx_n \to t \in M$.

For an equivalent formulation for two single-valued maps on a metric space refer [19-20]. We remark that commuting maps are weakly commuting and weakly commuting maps are compatible, and the reverse implication is not true (see, for instance, [19-20], [24], [34], [44-45] and [47-48]). However, all the three concepts for a pair of maps are equivalent at a concidence point of the maps.

Let $T : X \to CL(X)$ and $f : X \to X$. We define (T, f) to be:

hybrid contraction if

(1) $H(Tx, Ty) \leq kd(fx, fy)$ for some $0<k<1$, and x, y in X;

hybrid nonexpansive if

(2) $H(Tx, Ty) \leq d(fx, fy)$, x, y in X; and

hybrid contractive if

(3) $H(Tx, Ty) < d(fx, fy)$, $fx \quad fy$ and x, y in X.

These definitions are modeled respectively on Banach contraction, nonexpansive and Edelstein's contractive maps [9-10] when $T : X \to X$ and $f = id$. Maps satisfying variants of (1)-(3) will be called hybrid type contraction/nonexpansive/contractive. For example, T and f satisfing.

(4) $H(Tx, Ty) \leq k \max \{d(fx, fy),\ d(fx, Tx),\ d(fy, Ty),$
$(1/2)[d(fx, Ty) + d(fy, Tx)]\}, 0<k<1$ and $x, y \in X$,

will be called '**hybrid type contraction**'. If no distinction is needed in a particular situation, maps falling in either of the three or similar categories will generally be described as **hybrid contracting maps** (Note that (1) implies (i), $i = 2, 3, 4$.)

The study of hybrid contracting maps was initiated during 1981-83 by Bhaskaran-Subrahmanyam [3] (see also [43], p.3), Hadzic [12], Kaneko [21], Mukherjee [32], Naimpally et al. [35] and Singh-Kulshrestha [49]. Indeed, the following coincidene theorem has a constructive proof in [49].

Theorem 1

Let $T: X \to CL(X)$ and $f: X \to X$ be such that (4) obtains, $f(X)$ is a complete subspace of X and

(1.1) $T(X) \subset f(X)$.

Then T and f have a coincidence. Indeed, for any x_o in X, there exists a sequence $\{x_n\}$ in X such that

(i) $fx_{n+1} \in Tx_n$, $n = 0, 1, 2, \ldots;$

(ii) $\{fx_n\}$ converges to fz for some z in X, and $fz \in Tz$,

i.e., T and f have a coincidence at z; and

(iii) $d(fx_n, fz) \leq (k^{1-q})^n (1-k^{1-q})^{-1} d(fx_o, fx_1)$,

where $q < 1$ is any positive number.

Coincidence theorems for (T,f) satisfying (1) and (1.1) appear in [12], [21] and [32] under strong conditions (such as continuity, commutativity) on the maps, Extensions of Theorem 1 appear in [42-43] and [48]. We remark that the condition (1.1) is sufficient for the existence of a sequene $\{fx_n\}$ but not necessary. Theorem 1 with $f = id$ is an important generalization of the multivalued contraction principle [8, 23] due to 'Ciric' [6, Th.2]. Indeed, the condition (4) of Theorem 1 includes several contraction type conditions for a multivalued map T on X with values either in $CB(X)$ or $CL(X)$ studied among others by Covitz-Nadler [8], Iseki [15], Kaneko [22-23], Kita [26, p. 115]. Nadler [33], Ray [39], Reich [40] and Wegrzyk [57]. Further, using 'Ciric's result [op.cit.] and following [35, Th. 2], one may also prove that T and f under the conditions of Theorem 1 have a coincidence.

Example 1

Under the conditions of Theorem 1, one can not expect, in general, a common fixed point for the maps T and f. In fact, let $X = [0, \infty)$, $Tx = [1+x, \infty)$ and $fx = 2x$ for x in X (see [35], [43]). Then all the hypotheses of Theorem 1 are satisfied, and T, f are common fixed point free. Indeed, T has no fixed point in X.

Example 2

Condition (1.1), although sufficient, is not necessary to obtain a coincidence. Let $X = [0, \infty)$, $Tx = [0, e^x]$ and $fx = 2(e^x-1)$, then $f(X)$ is complete, (4) obtains, (1.1) fails and T and f have a coincidence, in fact, a common fixed point.

SOME HISTORICAL NOTES

The condition (1) with T single-valued on X was first studied by R. Machuca [29] in 1967 followed by Goebel [11] in 1968 (see (1*) below). In a symposium on Fixed Point Theory and its Applications (Meerut University, Meerut, 1982), papers of Bhaskaran-Subrahmanyam and Singh-Kulshrestha attempting to investigate conditions under which (T, f) satisfying (1)/(4) could have a common fixed point were a subject of discussion (see also [43], p. 3). Mukherjee [32] attempted, in vain, to establish a common fixed point theorem for continuous and commuting maps T and f satisfying (1) and (1.1) with X complete. Olga Hadzic [12] studied the following hybrid type contraction condition for $f, g : X \to X$ and $T : X \to CB(f(X) \cap g(X))$:

(5) $H(Tx, Ty) \leq kd(fx, gy),\ x,\ y \in X,$

and attempted a solution to $fx \in Tx$ and $gx \in Tx$, wherein f, g are continuous and T is closed. Hadzic's result is a special case of Theorem 4 (see below).

Let Y be an arbitrary set and $f, g \colon Y \to X$ such that for $0<k<1$ and x, y in Y,

(1*) $g(Y) \subset f(Y)$ and $d(gx, gy) \leq kd(fx, fy)$.

Machuca [op. cit.] attempted to show the existence of a coincidence point of f and g satisfying (1*) under heavy topological conditions. Goebel

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+In a chat (Thunder Bay, 1983), Prof, G. Jungck denied the knowledge of [29]and [11].

[op. cit.] using the Banach contraction principle proved that the maps f and g have a coincidence when (1*) obtains and $f(Y)$ is complete. In 1976, Jungck [17], perhaps unaware⁺ of Machuca [op. cit.] and Goebel [op. cit.], gave a constructive proof to show that continuous commuting maps f and g satisfying (*1**) with $Y=X$ complete have a unique common fixed point. This fixed point theorem together with [18] led to a massive growth of fixed point theorems for commuting, weakly commuting and compatible maps; we cite [2-4], [12-13], [19-21], [24], [28], [31-32], [34-35], [37-38] and [42-50] as examples.

A metric space X is metrically convex if for any distinct x, y in X, there exists z in X distinct from x,y such that $d(x,z) + d(z,y) = d(x,y)$. Let X be a complete and metrically convex metric space, and K a nonempty closed subset of X. Let $f\colon K \to X$ be a continuous map and S, $T\colon K \to CB(X)$ such that for all x,y in K,

$$\text{(BS) } H(Sx, Ty) \leq \alpha\, d(fx, fy) + \beta\, [d(fx,Sx) + d(fy, Ty)]$$

$$+\ \gamma\, [d(fx, Ty) + d(fy, Sx)]$$

Where $\alpha, \beta, \gamma \geq 0$ with $(\alpha + \beta + \gamma)(1 + \alpha + \gamma)/(1-\beta-\gamma)^2 < 1$. Assad-Kirk [1] studied (BS) with $S = T$, $f = id$ and $\beta = \gamma = 0$ by levying certain boundary conditions on the map T. Subsequently, Itoh [16], Khan [25], Tan-Minh [55], Yanagi [58] and others generalized their result [1, Th.1] for multivalued maps on K with values in $CB(X)$. Initial coincidence/fixed point theorems under (BS) and (BS) with $S = T$ obtained in [4] extend certain nice fixed point theorems of Itoh [op.cit.] and Khan [op. cit.].

Following Assad-Kirk [1], Bhaskaran-Subrahamanyam [4] (see also [3]) initiated the study of hybrid nonexpansive maps T and f on a subset of a banach space, and obtained coincidence/fixed point theorems for such maps. On the other hand, Hadzic' [13] initiated the study of hybrid nonexpansive maps and its variants on convex metric spaces (convexity in the sense of Takahashi, see [13]) and convex probabilistic metric spaces.

COINCIDENCE AND FIXED POINT THEOREMS FOR HYBRID TYPE CONTRACTIONS

Several generalizations, variants and special cases of Theorem 1 have been obtained on several settings (see, for instance, [21], [24], [28], [34],

[35], [37], [42], [43], [45], [48], [50] and [60-61]). Kaneko [21] proved that continuous and commuting maps $T:X \to CB(X)$ and $f:X \to X$ satisfying (1) and (1.1) have a coincidence in X provided X is complete. Generalizing this result Kaneko-Sessa [24] obtained the following :

Theorem 2

Let X be a complete metric space, and $f: X \to X$ and $T: X \to CB(X)$ be compatible continuous maps such that (1.1) and (4) hold. Then T and f have a coincidence.

We remark that the conditions of this theorem do not guarantee the existence of a common fixed point of T and f (see, for instance, Example 2 below). However, they [24, p. 260] (see also [21, Th. 1]) have shown that T and f with all the assumptions of Theorem 2 have a common fixed point whenever one of the following holds : either $fx \neq f^2x$ implies $fx \notin Tx$, or $fx \in Tx$ implies that $f^n x \to z$ for some $z \in X$. These conditions appear to be more acceptable than those cosmetized in [43].

In [50], Singh-Pant have obtained a coincidence theorem for two multivalued maps and a single-valued map in probabilistic analysis. However, the metric analogue of their result is a slightly improved by Theorem 3 (below), which is a special case of the main result of Singh-Ha-Cho [48].

Theorem 3

Let X be a metric space, $S, T:X \to CL(X)$ nd $f: X \to X$ such that $S(X) \cup T(X) \subset f(X)$. $f(X)$ is a complete subspace of X and

(6) $H(Sx, Ty) \leq k \max \{d(fx,fy), d(fx,Sx), d(fy,Ty), (1/2)[d(fx,Ty)+d(fy,Sx)]\}$

for some $0<k<1$ and all x,y in X. then S, T and f have a coincidence, i.e., there exists a point z in X such that $fz \in Sz \cap Tz$. Further, if for such a z, fz is a fixed point of f, then fz is also a fixed point of S (resp. T) provided f and S (resp. T) are weakly commuting at z, fz is a common fixed point of S and T provided f weakly commutes with each of S and T.

Indeed, in [48], the conclusions of Theorem 3 are obtained for conditions more general than those in Theorem 3. We remark that, as the following example shows, without the assumption "fz is a fixed point of f" (or without conditions leading to it) in Theorem 3, f, S and T need not have a common fixed point, even if the maps are continous and commuting on X. Moreover, "weakly" in Theorem 3 is superfluous, since, recall that, the weak commutativity at a coincidence point of T and f is equivalent to the commutativity at the coincidence point of T and f. Moreover, several results of [59] appear to be special cases of [48].

Example 2 [48]

Let $X = [0,1]$ and $Sx = Tx = \{0,1\}$, $fx = 1-x$ for all x in X. Then all the hypotheses of Theorem 3 are satisfied except that none of the coincidence values, viz., $f0$ or $f1$, is a fixed point of f. Note that f and S are continuous and commuting on X.

Coincidence and fixed point theorems for two multivalued and two single-valued maps on a metric space have been obtained among others by Kubiak [28], Naidu-Prasad [34], Naidu [61], Popa [37] and Sessa-Fisher [45]. Following Hadzic [12], Kubiak [28] established the following coincidence theorem.

Theorem 4 [28]

Let X be a complete metric space, f and g continuous maps from X into X, S and T closed maps from X into $CB(f(X) \cap g(X))$ such that $fSx = Sfx$, $gTx = Tgx$ for every x in X and

$$(7)\quad H(Sx,\ Ty) \leq k \max\{d(fx,\ gy), d(fx,\ Sx), d(gy,\ Ty), (1/2)[d(fx,\ Ty) + d(gy,\ Sx)]\}$$

for some $0<k<1$ and all x, y in X. Then there exists a point z in X such that $fz \in Sz$ and $gz \in Tz$.

Now, we have the following coincidence theorem for a sequence of multivalued maps, wherein N is the set of natural numbers.

Theorem 5

Let X be a complete metric space, f and g continuous maps from X into X and $\{Tn : n \in N\}$ a sequence of closed maps from X into $CL(f(X) \cap g(X))$ such that, for each n in N and each x is X, $fT_n x = T_n fx$, $gT_n x = T_n gx$, and

$$(8)\quad H(T_m x,\ T_n x) \leq$$

$k \max \{d(fx,gy), d(fx,T_m x), d(gy,T_n y), 1/2[d(fx,T_n y)+d(gy,T_m x)]\}$ for some $0<k<1$ and all x,y in X, $m \neq n$. Then there exists a point z in X such that $fz \in T_n z$ and $gz \in T_n z$ for each $n \in N$.

Proof

An appropriate blend of the proofs of Kita [26] and Kubaik [28] will yield the result.

Note that Theorem 5 with $T_{2n-1} = S$ and $T_{2n} = T$ for each n in N slightly improves Theorem 4. Following Pal-Maiti [36] and the proof of Theorem 1 (above) (see [49]) we have the following :

Theorem 6

Let X be a metric space, $f: X \to X$ and $T : X \to CL(X)$ such that (1.1) obtains and $f(X)$ is a complete subspace of X. Further, let T and f satisfy, for x,y in X, at least one* of the following :

$$H(fx,Tx)+H(fy,Ty) \leq a d(fx,fy), \quad 1<a<2;$$

$$H(fx,Tx)+H(fy,Ty) \leq b\ [d(fx,Ty)+d(fy,Tx)+d(fx,fy)], 1/2<b<2/3;$$

$$H(fx,Tx)+H(fy,Ty)+H(Tx,Ty) \leq c\ [d(fx,Ty)+d(fy,Tx)], 1< c <3/2;$$

and the condition (4) with $0<k<1$. Then there exists a point z in X such that $fz \in Tz$. Further, if fz is a fixed point of f, and if f and T commute at z, then fz is also a fixed point of T.

We remark that, in Theorem 6, if $f = id$ on X, then we get a result from [51], and if, in addition, T is a single-valued map on X, then we get an important fixed point theorem of Pal-Maiti [36].

As regards hybrid type contractions on a product space, a coincidence theorem has recently been obtained in [2] for two systems of multivalued maps and a system of single-valued maps on a finite product of metric spaces. It has been done by using a fixed matrix of non-negative numbers

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*The intended meaning is that, for each pair of points x, y of X, one (or a set of more than one) of the four conditions is to be satisfied.

(see [30]) and a new class of coordinatewise weakly commuting systems of single and multi-valued maps on the product space. It is too complicated to recall those results here. For an equivalent formulation of coordinatewise weakly commuting two sets of single-valued maps, refer [47].

HYBRID CONTRACTIVE MAPS

The following appears in [35].

Theorem 7

Let X be a metric space, $T : X \to CL(X)$ and $f : X \to X$ be such that $T(X) \subset f(X)$ and $f(X)$ is a compact subset of X. Suppose that for every $x, y \in X$,

$$(9) \qquad \begin{array}{l} H(Tx, Ty) < d(fx, fy), \text{ if } fx = fy \\ Tx = Ty \text{ if } fx = fy \end{array}$$

Then T and f have a coincidence.

For an attempt ;to obtain the conclusioin of this theorem under the strict inequality condition (4) with $k=1$, see [38]. Theorem 7 includes the following variant of Naimpally et al. [35, Cor. 3] (see also [11, p. 734]).

Theorem 8

Let X be a metric space and $f, g : X \to X$ be such that $g(X) \subset f(X)$, $f(X)$ is compact and

$$(10) \qquad d(gx, gy) < d(fx, fy), fx \neq fy$$

for x, y in X. Then f and g have a coincidence. Further, if f and g commute at their coincidence points, then f and g have a unique common fixed point.

Theorem 8 includes certain important fixed point theorems of Edelstein [9-10] and Jungck [18]. For more general results, refer [20]. If $X = R^n$ we have the following improved form of Theorem 8.

Theorem 9 [46]

Let $f, g : R^n \to R^n$ be such that $g(R^n) \subset f(R^n)$, g is bounded and

$$|| gx - gy || < || fx - fy || \quad , fx \neq fy$$

for x, y in R^n. Then f and g have a coincidence. Further, if f and g commute at their coincidence points, then f and g have a unique common fixed point.

Theorem 9 with $f = id$ on R^n is an important result of Bridges et al. [5, Th. 2.2], since a contractive map of the closed unit ball in a Banach space need not have a fixed point (see [56] and also [53, p. 39]).

One may ask if Theorem 9 can be extended to the case when g is a multivalued map. In particular, we have the following :

Question : Let $T : R^n \to CL(R^n)$ and $f : R^n \to R^n$ such that $T(R^n) \subset f(R^n)$, $f(R^n)$ is bounded and (9) holds for $x, y \in R^n$. Will T and f have a coincidence ?

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 207-212

APPLICATION OF MODERN ASTRONOMY TO CERTAIN PROBLEMS OF HINDU ASTRONOMY†

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(Received 03-12-1994)

ABSTRACT

Hindu methods of computing planetary positions of Sun, Moon etc. and calculating *Mandaphala* (Equation of Centre) and *Gatiphal* are some what lengthy. In this paper, we have suggested certain techniques of Modern Astronomy to compute planetary position, *Manadaphala* and *Gatiphala*. The methods are explained by taking examples from Hindu Astronomy.

Mathematics Subject Classification (1991) : 01A32

Keywords and Phrases : *Spars'a-Sthiti Khaṇḍa, Moksa-Sthiti-Khaṇḍa; Sammilana-Marda-Khaṇḍa, Unmilana-Marda-Khaṇḍa.*

INTRODUCTION

It appears that *Bhāskarācārya* II (b.1114 A.D.) had notions of Differential Calculus, but certainly not in the modern language. The above idea, 500 years before Newton and Leibntz, appears in his famous treatise *Siddhānta Siromaṇi* (SS). For the exact computation of the daily motion of planets Bhāskara II introduced the concept of *Tatakālika Gati* (instantaneous velocity) by dividing the day into a large number of small intervals and compared the positions of planets at the end of successive intervals. If y and y′ are the mean anomalies of the planets at the end of consecutive intervals, then according to him

$Hsin\, y' - Hsin\, y = (y' - y)\, Hcos\, y$

(Here *Hsin* stands for Hindu-sine).

This result is equivalent to

$\delta(sin\, y) = Cos\, y\, \delta y$

(see, [3, p.110], [4, p. 146-148] and [1, p.92]).

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† Major part of this work was presented at the symposium on Science nad Technology in Ancient India, held at B.M. Birla Science Centre Hyderabad Dec. 1-2, 1990.

Professor Srinivasiengar [op.cit.] observes that *Bhāskara* has gone deeper into Differential Calculus in *Siddhānta Śiromaṇi* and suggests that the differential coefficient vanishes at an extreme value of the function. In the case of lunar and solar eclipses, he says that celestial latitude (β) of the moon plays an important role for the determination of *Sparsa-Sthiti-Khaṇḍa* (the time between the moment of first contact and the middle of eclipse, i.e. half the duration of an eclipse), *Mokṣa-Sthiti-Khaṇḍa* (the time between the middle of the eclipse and the moment of the last contact), *Sammilana-Marda-Khaṇḍa* (the time between commencement of total eclipse and the middle of the eclipse) and *Unmilana-Marda-Khaṇḍa* (the time between the middle of the total eclipse and the end of the eclipse). It is well-known that celestial latitude (β) of the moon varies from moment to moment. For the exact computation of celestial latitude (β) at a point of the orbit of the moon, *Bhāskara* introduced a method in the following two verses of the SS :

स्थित्यर्धनाडीगुणिता स्वभुक्तिः
षष्ट्या ६० हृता तद्रहितौ युतौ च ।
कृत्वेन्दुपा[illegible]सकृच्छराभ्यां
स्थित्यर्धमाद्यं स्फुटमन्तिमं च ॥
एवं विमदार्धफलोनयुक्त-
सपातचन्द्रोद्‌भवसायकाभ्याम् ।
पृथक् पृथक् पूर्ववदेवसिद्धे
स्फुट स्त आद्यन्त्यविमर्दखण्डे ॥

(The first verse gives the rectification of the times of *Sparsa-Sthiti-Khaṇḍa* and *Mokṣa-Sthiti-Khaṇḍa*, and the second verse gives the rectification of *Sammilana-Madra-Khaṇḍa* and *Unmilana-Marda-Khaṇḍa*).

However, when we follow the *Bhāskara's* method to calculate (β), we have to undergo somewhat lengthy calculations. Since the basic ideas of calculus are woven in the theory, we can apply modern calculus. For example to get the rectified value of β, let us differentiate the following with respect to β:

$H\sin\beta = (H\sin\lambda \; H\sin i)/R$

to get

$H\cos\beta\delta\beta = (H\sin i \; H\cos\lambda\delta\lambda)/R$

wherein i is the inclination of moon's orbit to ecliptic i.e., $i = 5°8'40''$, $R = 3438'$ and $(H\sin i)/R$ is a constant and λ = Moon's longitude - Node's longitude.

It may be mentioned that this formula gives in one stroke the rectified celestial latitude of the moon at the respectdive moments from which the

respective rectified time may be computed immediately. The calculus may also be applied in several other situations.
We illustrate the method by an example.

Example : Compute the *Sparsa-Sthiti-Khaṇda* and *Mokṣa-Sthiti-Khanda* of partial lunar eclipse on August 6, 1990 at Hardwar.

Step 1

First we get the rectified planetary positions after making necessary corrections of *Desāntara, Carajyā*, *Bhujāntara* and *Udayāntara* of the sun, the moon and the lunar node with the help of *ahargaṇa.*

The following table gives the desired planetary position on August 6, 1990 at Hardwar.

Rectified longitude of the sun (Rls)	3^R 20° 25' 49"
Rectified longitude of the moon (Rls)	9^R 15° 35' 19"
Longitude of the lunar node	9^R 12° 59' 58"
Difference of the sun and the lunar node	282° 59'58"-110° 25'45"=172°34'13"
Distance of the sun from the lunar node	180°-172° 34'13"=7° 25'47" < 9°
Rlm-Rls	9^R15°35'19"-3^R20°25'45"=5^R25°9'34" =175° 9' 34"
Difference to cover 180°	180°-175° 9' 34"=4°50' 26"=17426"
Rectified motion (*Gati*) of the sun (s)	58' 18" 10'''
Rectified motion (*Gati*) of the moon	765' 1"
m-s	765' 1"-58' 18" 10''' = 42463"
Duration of *Purnima* beginning from the sun rise i.e. from 0548 hrs (IST)	17426 x 60/42463 Ghatis 24.62284812 Ghatis 24.62284812×24 ÷ 60 hrs 9.849139246 hrs 9^h 50^m 56^s

Evidently there is a lunar eclipse on Aug. 6, 1990.
Note that Rlm - Rls = 7° 25' 47" < 9°

Step II:

Rectification of the sun, the moon and the lunar node at the end of *Purṇimā*

The following table shows the rectified longitudes of the sun, the moon and the lunar node at the end of *Purṇimā* (On Aug. 6, 1990).

Sun's motion during 60 *Ghatis*	$57' 18'' 10''' = 57.302777'$
Sun's motion during 24.62284812 *Ghatis*	$24.62284812 \times 57.302777/60 = 23'30''$
Rectified longitudeof the sun at the end of *Purṇimā*	$3^R 20^\circ 25' 45'' + 23'30'' = 3^R 20^\circ 49' 15''$
Moon's motion during 24.62284812 *Ghatis*	$24.62284812 \times 765.01/60 = 5^\circ 13'56''$
Rectified longitude of the moon at the end of *Purṇimā*	$9^R 15^\circ 35' + 5^\circ 13'56'' = 9^R 20^\circ 49' 15''$
Mean daily motion of the lunar node	$0\text{-}3'\text{-}10''\text{-}48''\text{-}20''' = 3' 11'''$ approx.
Increment of the lunar node	$24.62284812 \times 3.180092592/60 = 1'18''$
Rectified longitude of the lunar node	$9^R 12^\circ 59' 58'' - 1' 18'' = 9^R 12^\circ 58' 40''$

ρ = Angular radius of the shadow cone $= (2m/15-5s/12)'/2$
$= (2 \times 765.01/15 - 5 \times 57.3027/12)'/12$
$= 39.0626'$

r = Angular radius of the moon $= 3m'/148 = 15.50'$

Moon's longitude with respect to the lunar node $= 9^R 20^\circ 49' 15'' - 9^R 12^\circ 58'40''$
$= 7^\circ 50' 35''$

Celestial latitude (β) of the moon = (Hsin λ × Hsin i)/3438
$= 3438 \sin 7^\circ 50'35'' \times 270' \div 3438 = 36.38''$

Now $\rho + r - \beta = 17.93'$

CONDITION FOR LUNAR ECLIPSE

Total lunar eclipse occurs when $\rho+r-\beta>2r$ and a partial lunar eclipse occurs when $\rho+r-\beta<2r$ since $\rho+r-\beta=17.39'<31'$. There is a partial lunar eclipse on Aug. 6, 1990.

Sparsā-Sthiti-Khaṇḍa $= [(\rho+r)^2-\beta^2]^{1/2}\times 60\div m\text{-}s$ *Ghatis*
(The first half duration of the eclipse) $= 3.427754943$ *Ghatis*

Therefore the length of the lunar eclipse is twice this period, i.e. *6.85550986 Ghatis*. This value is further refined 2 hrs. and 44.5 minutes.

Step III

The Following *Bhāskara II* (cf. second verse above), we get the following:

$$\lambda_1 = 7^\circ\ 6'\ 33' \quad \text{and} \quad \lambda_2 = 8^\circ\ 34'\ 27''$$

Wherein λ_1 (respectively λ_2) is the difference of the longitude of the moon and its node at the first (respectively last point of contact).

$$\text{Also } \beta_1 = 33.372' \quad \text{and} \quad \beta_2 = 39.90'$$

where β_1 and β_2 are the rectified values of β at the beginning and end of the eclipse. These vlaues may be calculated in a manner analogeous to β. (Further rectification of β_2 gives the same value).

STEP IV

Refined-*Sparsa-Sthiti-Khaṇḍa* $= [(\rho+r)^2-\beta_1^{\,2}]^{1/2}\times 60\div m\text{-}s$
$= 3.699716455$ *Ghatis*
$= 1$ *hr* 27 *m* 49 *sec.*

Mokṣa-Sthiti-Khaṇḍa $= \left[(\rho+r)^2-\beta_2^2\right]^{1/2}\times 60\div m-s$
$= 3.155232422$ *Ghatis*
$= 1$ *hr* 15 *m* 43 *sec.*

Now we apply modern calculus to get the rectified value of celestial latitude of the moon, we use the calculations of step I-III. Differentiating the formula :

$$H\sin\beta = 270'\times H\sin\lambda \div 3438\ Kal\bar{a}$$

we get

$$\delta\beta = 270'\times \cos\lambda\,\delta\lambda \div 3438\cos\beta\ Kal\bar{a}$$

$$= 270' \cos 7^\circ 6' 53'' \times 1^\circ 27' 44'' \div 3438 \cos 36.36' \text{ } Kal\bar{a}$$

$$= 6.48 \text{ } Kal\bar{a}$$

In fact this is the approximate average change in the celestial latitude of the moon.

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फार्म - ४

प्राकृतिक एवं भौतिकीय विज्ञानशोध पत्रिका

सम्मिलित खण्ड ५-८ १९९१-१९९४

(१) प्रकाशन स्थान - गुरुकुल कांगड़ी विश्वविद्यालय, हरिद्वार
(२) प्रकाशन की अवधि - वर्ष में एक खण्ड अधिकतम दो अंक किन्तु यह सम्मिलित खण्ड है
(३) मुद्रक का नाम - अवधेश शिवपुरी
सद्भावना प्रिण्टर्स एण्ड एलाइड़ ट्रेडर्स
राष्ट्रीयता - भारतीय
व - एफ २२ औद्योगिक क्षेत्र, हरिद्वार
पता ✆ – ४२५७५१
(४) प्रकाशक का नाम - डा० जयदेव वेदलंकार
राष्ट्रीयता - भारतीय
व पता कुलसचिव गुरुकुल कांगडी विश्वविद्यालय हरिद्वार -२४९४०४
(२) प्रधान सम्पादक - डा० एस० एल० सिंह
राष्ट्रीयता - भारतीय
व पता - गणित विभाग, गुरुकुल कांगडी विश्वविद्यालय हरिद्वार - २४९४०४
(६) प्रबन्ध सम्पादक - डा० पी० पी० पाठक
राष्ट्रीयता - भारतीय
व पता - भौतिकी विभाग, गुरुकुल कांगडी विश्वविद्यालय, हरिद्वार
(७) स्वामित्व - गुरुकुल कांगडी विश्वविद्यालय, हरिद्वार-२४९४०४

मैं, जयदेव वेदालंकार, कुलसचिव गुरुकुल कांगडी विश्वविद्यालय हरिद्वार घोषित करता हूँ कि उपरिलिखित तथ्य मेरी जानकारी के अनुसार सही हैं।

हस्ताक्षर
जयदेव वेदालंकार
कुलसचिव

Volume 9-10 (1995-96)

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका

JOURNAL OF NATURAL & PHYSICAL SCIENCES

गुरुकुल कांगड़ी विश्वविद्यालय, हरिद्वार
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Journal of Natural & Physical Science Vol. 9-10 (1995-96) pp 1-12

ON AN ABSTRACT FUNCTIONAL INTEGRODIFFERENTIAL EQUATION

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(Received 07-03-95)

ABSTRACT

The aim of the present paper is to study the existence of a unique weak solution of an abstract functional integrodifferential equation of the more general type by using the well known Banach fixed point theorem and the integral inequality established by Panchpatte.

INTRODUCTION

Let X be a general Banach space. Let $C = C([-r, 0], X)$, $0 < r < \infty$, denotes the space of continuous functions mapping $[-r, 0]$ into X with norm $\|\phi\|_C = Sup_{\theta\in[-r.\ 0],} \|\phi\theta\|$, for $\phi\theta\in C$. If x is a continuous function from $[-r,T]$, $(T>0)$ to X and $t\in[0, T]$, then x_t denotes the element of C given by $x_t(\theta) = x(t+\theta)$ for $\theta\in[-r, 0]$. In the present paper, we study the following abstract functional integrodifferential equation of the type.

$$x'(t) + A(t)x(t) = f(t, x_t, \int_0^t k(t,s,x_s)ds) \tag{1.1}$$

$$t\in[0, T],$$

$$x_0 = \phi[t], \quad -r \le t \le t\ 0,$$

where $x : [-r, T] \to X$, $A(t) : D(A(t)) \subset X \to X$, the functions $k : [0, T] \times [0, T] \times C \to X$ and $f : [0, T] \times C \times X \to X$ are continuous.

The equations of this type or its special forms commonly come across in almost all areas of applied Mathematics, see [1,2] and references listed therein. The problems of existence, uniqueness and other

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properties of the solution of the equation (1.1) or its special forms have been studied by many authors in the literatute by using different techniques, see [3-10, 12]. In particular, Kartsators and Parrott [9,10] have proved the existence of a unique weak solution of equation (1.1) when $k = 0$. In a recent paper [4], Dhakne and Pachpatte have investigated an approximation scheme for the special form of the equation (1.1) with X' uniformly convex, see also [5,6].

The main purpose of the present paper is to study the existence and stability of the weak solution of the equation (1.1) by using the well known Banach fixed point theorem and the integral inequality established by Pachpatte [11].

PRELIMINARIES AND STATEMENT OF RESULTS

Before proceeding to the statements of our main results, we shall setforth some preliminaries from [9] and hypotheses on the functions involved in (1.1) that will be used in our subsequent discussion.

An operator $A : D \subset X \to X$ is said to be m-accretive if

$$\| x - y + \lambda (Ax - Ay) \| \geq \| x - y \|$$

for every $x, y \in D$, $\lambda > 0$ and $R(I + \lambda A(t)) = X$.

By a weak solution $\bar{x}(t)$ of the equation (1.1) on $[-r, T]$, we mean for $t \in [-r, 0]$, $\bar{x}(t) = \phi(t)$, for $t \in [0, T]$, $\bar{x}(t) = U(t, 0)\phi(0)$, where $\{U(t,s) : 0 \leq s \leq t \leq T\}$ is a family of evolution operators on $[0, T]$ for the problem,

$$(2.1) \qquad t \in [0, T], \qquad x(0) = \phi(0)$$

(See Evans [8]).

We list the following hypotheses for convenience.

(H_1) $A(t)$ is m-acdretive for each $t \in [0, T]$.

(H_2) There exists a constant $\lambda_0 > 0$, a continuous nondecreasing function $r_1 : [0, \infty) \to [0, \infty)$ and a continuous function $p : [0, T] \to X$ such that for all $\lambda \in (0, \lambda_0)$, $t, s \in [0, T]$ and $x \in \overline{D(A(t))}$

$$\|(I + \lambda A(t))^{-1} x - (I + \lambda A(s))^{-1} x\| \le \|p(t) - p(s)\| \, r_1(\|x\|). \tag{2.2}$$

(H_3) There exist continuous functions $m : [0, T] \to [0, \infty)$, $q : [0, T] \times [0, T] \to X$ and a continuous nondecreasing function $r_2 : [0, \infty) \to [0, \infty)$ such that

$$\| k(t,s,\phi) - k(t,s,\psi) \| \le m(s) \| \phi - \psi \|_C \tag{2.3}$$

$$\| k(t_1,s_1,\phi) - k(t_2,s_2,\phi) \| \le \| q(t_1,s_1) - q(t_2,s_2) \| r_2(\|\phi\|_C) \tag{2,4}$$

for all $\phi, \psi \in C, t, t_1, t_2, s, s_1, s_2, \in [0, T]$ and

$$\int_0^\infty m(t)dt < \infty \tag{2.5}$$

(H_4) There exist continuous functions $n : [0, T] \to [0, \infty)$, $h : [0, T] \to X$ and a continuous nondecreasing function $r_3 : [0, \infty) \times [0, \infty) \to [0, \infty)$ such that

$$\| f(t,\phi,x) - f(t,\psi,y) \| \le n(t) [\| \phi - \psi \|_C + \| x - y \|] \tag{2.6}$$

$$\| f(t,\phi,x) - f(t,\phi,x) \| \le \| h(t) - h(s) \| \, r_3(\| \phi \|_C, \| x \|) \tag{2.7}$$

for all $\phi, \psi \in C$, $t, s \in [0, T]$ $x, y \in X$ and

$$\int_0^\infty n(t) e^{-\alpha t} dt < \infty, \alpha < 0. \tag{2.8}$$

(H_5) The initial function $\phi \in C$ and $\phi(0) \in \overline{D(A(0))}$

REMARK 1 : It is to be noted that the hypotheses (H_1) and (H_2) imply that the set $\overline{D(A(t))}$ is independent of t. (See [8], Lemma 3.1).

We state the following lemma established by Pachpatte in [11] which is effectively employed in the proof of our main result.

Lemma : (See pachpatte [11, p. 758]). Let $a(t)$, $b(t)$ and $c(t)$ be real valued nonnegative continuous functions defined on R^+, for which the inequality

$$c(t) \le c_0 + \int_0^t a(s)c(s)ds + \int_0^t a(s)[\int_0^s b(\tau)c(\tau)d\tau]ds,$$

holds for all $t \in R^+$, where c_0 is a nonnegative constant. Then

$$c(t) \le c_0[1 + \int_0^t a(s)\exp\{\int_0^s (a(\tau)+b(\tau))d\tau\}ds]$$

for all $t \in R^+$

Our main results are established in the following theorems.

THEOREM 1. Suppose that the hypotheses (H_1) - (H_5) hold. Then there exists a unique weak solution $\bar{x}(t)$ of the equation (1.1) on $[-r, T]$.

In the following theorem, we assume that solutions of the equation (1.1) exist on $[-r, \infty)$.

THEOREM 2. Suppose that $A(t) + \alpha I$ is m-accretive for all $t \in [0, \infty)$, where α is a fixed negative constant. Further, suppose that hypotheses (H_2) - (H_5) are fulfilled with $[0, T]$ replaced by $[0, \infty)$. If $\bar{x}(t)$ and $\bar{y}(t)$ are the weak solutions of (1.1) and

$$y'(t) + A(t)y(t) = f(t, y_t, \int_0^t k(t, s, y_s)ds), t \in [0, \infty),$$

(2.9) $$y_0 = \psi(t), \qquad -r \le t \le 0,$$

respectively on $[-r, \infty)$, then the estimate

$$\|\bar{x}(t) - \bar{y}(t)\| \le L\|\phi - \psi\|_C \, e^{\alpha t}$$

where L is suitable constant, holds for all $t \in [0, \infty)$.

Remark 2. We note that in [12], Pachpatte has established several interesting results regarding the boundedness, stability and asymptotic behaviour of solutions of the equation (1.1) without functional argument when $A(t)$, for each $t \in R^+$ is linear closed operator by using comparision principle and the integral inequality of the more general type investigated by himself in [11]. Here our conditions on functions involved in (1.1) and the approach to the problem are different.

PROOFS OF THEOREMS 1 AND 2

The nonnegative functions m and n being continuous on compact interval $[0, T]$, there exist positive constants M and N such that $m(t) \leq M$ and $n(t) \leq N$ for $t \in [0, T]$. Let $S = \{u \in C([-r, T], X);\ u(t) = \phi(t)$ for $t \in [-r, 0]\}$. For $u \in S$, define $\| u \|_B = sup_{t\in[-r, T]} e^{\beta t} \| u(t) \|$, where β is a fixed negative constant such that $\beta^2 > N(2+MT)\,|\beta| + 2NM$. It is apparent that S is a complete metric space with this Bielecki norm. Let $u \in S$ and define $T : S \rightarrow S$ by

$$Tu(t) = \begin{cases} \phi(t)\ for\ t \in [-r, 0], \\ x_u(t)\ for\ t \in [0, t], \end{cases}$$

where $x_u(t)$ is the unique weak continuous solution of the equation

$$x'(t) + A(t)x(t) = f(t, u_t, \int_0^t k(t, s, u_s)ds), t \in [0, T],$$

(3.1) $\qquad x(0) = \phi(0)$

on $[0, T]$. The existence of such a solution is guaranteed by Evans [8, Theorem 1]. We now, show that T is a contraction operator on S.

Consider the following equation

$$x'(t) = A(t)x(t) = f(t, v_t, \int_0^t k(t, s, v_s)ds), t \in [0, T],$$

(3.2) $\qquad x(0) = \phi(0)$

where $v \in S$. Clearly, this aquation (3.2) has the unique strongly continuous weak solution $x_v(t)$ on $[0, T]$. We note that for $t \in [-r, 0]$, $Tu(t) = Tv(t)$. By making use of [8, Theorem 3], (2.3) and (2.6) we have, for $t \in [0, T]$, (3.3) $$||Tu(t) - Tv(t)|| = ||x_u(t) - x_v(t)||$$

$$\leq \int_0^t ||f(s, u_s, \int_0^s k(s,\tau,u_\tau)d\tau) - f(s, v_s, \int_0^s k(s,\tau,v_\tau)d\tau|| ds$$

$$\leq N \int_0^t ||u_s - v_s||_C\, ds + NM \int_0^t \int_0^s ||u_\tau - v_\tau||_C\, d\tau ds$$

which yields,

(3.4) $$e^{\beta t}||x_u(t) - x_v(t)||$$

$$\leq N e^{\beta t} \int_0^t ||u_s - v_s||_C\, as \; + MN e^{\beta t} \int_0^t \int_0^s ||u_\tau - v_\tau||_C\, d\tau ds$$

$$= N \int_0^t e^{\beta(t-s)} e^{\beta s} ||u_s - v_s||_C\, ds$$

$$+ MN \int_0^t \int_0^s e^{\beta(t-\tau)} e^{\beta\tau} ||u_\tau - v_\tau||_C\, d\tau\, ds$$

Now, we have

(3.5) $$e^{\beta s}||u_s - v_s||_C = e^{\beta s} sup_{\theta \in [-r,0]} ||u(s+\theta) - v(s+\theta)||$$

$$= sup_{\theta \in [-r,0]} ||e^{-\beta\theta} e^{\beta(s+\theta)}[u(s+\theta) - v(s+\theta)]||$$

$$\leq sup_{s^* \in [-r,T]} ||e^{\beta s^*}[u(s^*) - v(s^*)]||$$

$$= ||u - v||_B$$

using (3.3), (3.4) and (3.5), we have for $t \in [0,T]$,

$$e^{\beta t} ||Tu(t) - Tv(t)||$$

$$\leq N \int_0^t e^{\beta(t-s)} ||u - v||_B\, ds + NM \int_0^t \int_0^s e^{\beta(t-\tau)} ||u - v||_B\, d\tau ds$$

$$\leq \frac{2N}{||\beta||} ||u - v||_B + \frac{NM(2 + |\beta| T)}{|\beta|^2} ||u - v||_B$$

$$= \frac{(2N + NMT)|\beta| + 2NM}{|\beta|^2} \|u - v\|_B$$

$$\leq \|u - v\|_B$$

Hence, we get

$$\|Tu - Tv\|_B \quad \leq \quad \|u - v\|_B \tag{3.6}$$

and consequently T is a contraction. Therefore, T has a unique fixed point, say $\bar{x}$. Then, $\bar{x}$ is the unique weak solution of the equation (1.1) on [-r,T]. This completes the proof of the Theorem 1.

Let $\bar{x}$(t) and $\bar{y}$(t) be the weak solutions of the following equations

$$x'(t) + A(t)x(t) = f(t, x_t, \int_0^t k(t, s, x_s)ds), \qquad t \geq 0,$$

$$x_0 = \phi(t), -r \leq t \leq 0$$

and
$$y'(t) + A(t)y(t) = f(t, y_t, \int_0^t k(t, s, y_s)ds), \qquad t \geq 0,$$

$$y_0 = \psi(t), -r \leq t \leq 0,$$

respectively on *[-r, ∞]* obtained as in Theorem 1. Using the m-accretiveness of $A(t) + \alpha I$, the proofs of theorems 1 & 3 of Evans [8], (2.3) and (2.6) we get the inequality for $t \geq 0$.

$$\|\bar{x}(t) - \bar{y}(t)\| \quad \leq e^{\alpha t} \|\phi(0) - \psi(0)\|$$

$$+ \int_0^t e^{\alpha(t-s)} \|f(s, \bar{x}_s, \int_0^s k(s, \tau, \bar{x}_\tau)d\tau) - f(s, \bar{y}_s, \int_0^s k(s, \tau, \bar{y}_\tau)d\tau)\| ds$$

$$\leq e^{\alpha t}\|\phi(0) - \psi(0)\| + \int_0^t n(s)e^{\alpha(t-s)}\|\bar{x}_s - \bar{y}_s\|_C \, ds$$

$$+ \int_0^t n(s)e^{\alpha(t-s)} \int_0^s m(\tau)\|\bar{x}_\tau - \bar{y}_\tau\|_C \, d\tau \, ds$$

$$\le e^{\alpha t}\|\phi-\psi\|_C + e^{\alpha t}\int_0^t n(s)e^{-\alpha s}e^{-\alpha s}\|\bar{x}_s-\bar{y}_s\|_C\,ds$$

$$+\,e^{\alpha t}\int_0^t n(s)e^{-\alpha s}\int_0^s m(\tau)e^{-\alpha\tau}\|\bar{x}_\tau-\bar{y}_\tau\|_C\,d\tau\,ds$$

which yields

$$(3.7)\qquad e^{-\alpha t}\|\bar{x}(t)-\bar{y}(t)\|$$

$$\le \|\phi-\psi\|_C \int_0^t n(s)e^{-\alpha s}e^{-\alpha s}\|\bar{x}_s-\bar{y}_s\|_C\,ds$$

$$+\int_0^t n(s)e^{-\alpha s}\int_0^s m(\tau)e^{-\alpha\tau}\|\bar{x}_\tau-\bar{y}_\tau\|_C\,d\tau\,ds$$

Case 1 : Suppose $t \ge r$. Then for every $\theta \in [-r,0]$, we have $t+\theta \ge 0$. For each θ's, from (3.7), we obtain

$$e^{-\alpha(t+\theta)}\|\bar{x}(t+\theta)-\bar{y}(t+\theta)\|$$

$$\le \|\phi-\psi\|_C + \int_0^{t+\theta} n(s)e^{-\alpha s}e^{-\alpha s}\|\bar{x}_s-\bar{y}_s\|_C\,ds$$

$$+\int_0^{t+\theta} n(s)e^{-\alpha s}\int_0^s m(\tau)e^{-\alpha\tau}\|\bar{x}_\tau-\bar{y}_\tau\|_C\,d\tau ds$$

$$\le \|\phi-\psi\|_C + \int_0^t n(s)e^{-\alpha s}e^{-\alpha s}\|\bar{x}_s-\bar{y}_s\|_C\,ds$$

$$+\int_0^t n(s)e^{-\alpha s}\int_0^s m(\tau)e^{-\alpha\tau}\|\bar{x}_\tau-\bar{y}_\tau\|_C\,d\tau ds$$

which yields

$$(3.8)\quad e^{-\alpha t}\|\bar{x}_t-\bar{y}_t\|_C \le \|\phi-\psi\|_C + \int_0^t n(s)e^{-\alpha s}\|\bar{x}_s-\bar{y}_s\|_C\,ds$$

$$+\int_0^t n(s)e^{-\alpha s}\int_0^s m(\tau)e^{-\alpha\tau}\;\|\bar{x}_\tau-\bar{y}_\tau\|_C\,d\tau\,ds$$

Case 2 : Suppose $0 \le t < r$. Then for all $\theta \in [-r,-t]$, we have $t+\theta<0$. For such θ's, we have

$$e^{-\alpha(t+\theta)}\|\bar{x}(t+\theta)-\bar{y}(t+\theta)\|$$

$$= e^{-\alpha(t+\theta)}\|\phi(t+\theta)-\psi(t+\theta)\| \le \|\phi-\psi\|_C$$

and hence, we get,

(3.9) $$\|\overline{x_t}-\overline{y_t}\|_C \le \|\phi-\psi\|_C$$

since $t<r, -\alpha<0, \quad e^{-\alpha t}<e^{-\alpha r}$.

Therefore, from (3.9), we obtain

(3.10) $$e^{-\alpha t}\|\overline{x_t}-\overline{y_t}\|_C \le e^{-\alpha r}\|\phi-\psi\|_C$$

for $\theta- \in [-t,0], t+\theta \ge 0$. Then from (3.7) we get, as in the first case,

(3.11) $$e^{-\alpha t}\|\bar{x}_t-\bar{y}_t\|_C$$

$$\le \|\phi-\psi\|_C + \int_0^t n(s)e^{-\alpha s}e^{-\alpha s}\|\overline{x_s}-\overline{y_s}\|_C\, ds$$

$$+ \int_0^t n(s)e^{-\alpha s}\int_0^s m(\alpha)e^{-\alpha\tau}\|\bar{x}_\tau-\bar{y}_\tau\|_C\, d\tau\, ds$$

Thus, for every $t \in [0,\infty]$, we have from (3.8), (3.10), (3.11)

(3.12) $$e^{-\alpha t}\|\overline{x_t}-\overline{y_t}\|_C$$

$$\le e^{-\alpha r}\|\phi-\psi\|_C + \int_0^t n(s)e^{-\alpha s}e^{-\alpha s}\|\bar{x}_s-\bar{y}_s\|_C\, ds$$

$$+ \int_0^t n(s)e^{-\alpha s}\int_0^s m(\alpha)e^{-\alpha\tau}\|\bar{x}_\tau-\bar{y}_\tau\|_C\, d\tau\, ds$$

By an application of Lema with $C(t)$ $e^{-\alpha t}\|\bar{x}_t-\bar{y}_t\|_C$,we get from 3.12

$$(3.13)\qquad e^{-\alpha t}\|\overline{x}_t - \overline{y}_t\|_C$$

$$\le e^{-\alpha r}\|\theta - \Psi\|\left[1 + \int_0 n(s)e^{-\alpha s}\exp\left\{\int_0^{s}(n(\tau)e^{-\alpha\tau} + m(\tau))d\tau\right\}ds\right]$$

From (3.13), (2.5) and (2.8), it follows that

$$\|\overline{x}_t - \overline{y}_t\|_C \le \mathrm{L}\|\theta - \psi\|_C\, e^{\alpha t}$$

where L is suitable positive constant. Since

$$\|\overline{x}(t) - \overline{y}(t)\|_C \le \|\overline{x}_t - \overline{y}_t\|_C$$

$$\|\overline{x}(t) - \overline{y}(t)\| \le \mathrm{L}\|\phi - \psi\|_C\, e^{\alpha t}$$

Hence, the solution $\overline{x}(t)$ of (1.1) is asymptotically stable and this completes the proof of the Theorem 2.

FURTHER APPLICATIONS

In order to illustrate the applications of our main results, we consider the following equation

$$(4.1)\qquad v'(\varsigma) = H(v(\varsigma)) + F(v(\tau\varsigma)), \int_l^{\varsigma} k(v(\varsigma\eta))d\eta$$

where $H, K\colon X \longrightarrow X$ *and* $F\colon X \times X \longrightarrow X$ are nonlinear operators and λ is a constant in (0,1).

Now consider the transformation

$\varsigma = e^{t}$ *and set* $\tau = e^{-r}, \eta = e^{s}, v(\varsigma) = u(t)$ in the equation (4.1) to obtain the equation

$$(4.2)\qquad u'(t) = e^{t}H(u(t)) + e^{t}F(u(t-r), \int_0 e^{s}k(u(s-r))ds)$$

Assume the following conditions

1. There exists $\alpha < 0$ such that -H+α isaccretive and $-R(I - \mu H) = x$ *for* $\mu > 0$.
2. *K* is a Lipschitz continuous everywhere defined operator with

Lipschitz norm β.

3. F is a Lipschitz continuous in either arguments everywhere defined operator with Lipschitz norm.
4. There is an increasing function $J:[0,\infty) \rightarrow [0,\infty)$ such that $\|H(u)\| \leq (\|u\|)$.

By using the methods [7] and [3, Lemma 3.2], it is easy to observe from conditions (1) - (4) that the hypotheses (H_1) - (H_4) hold. Now, an application of Theorem 1 yields a unique weak solution u(t) of the equation (4.2), $u_0 = \phi$, and for any $\phi \in C$ with $\phi(0) \in \overline{D(H)}$ on *[-r,T]*. Since $T \in (0,\infty)$ can be chosen arbitrarily large, this solution *u(t)* is acutally extendable ro $[-r,\infty]$. By an application of Theorem 2, the solution *u(t)* of (4.2), $u_0 - \phi$, (where ϕ is as above), is asymptotically stabe.

ACKNOWLEDGEMENT

The author expresses his sincere gratitude to Professor B.G. Pachpatte for many helpful discussions and suggestions during the preperation of this paper.

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Journal of Natural & Physical Science Vol. 9-10 (1995-96) pp 13-24

CONVERGENCE OF SEQUENCES OF ITERATES OF MULTIVALUED OPERATORS[#]

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(Received 17-07-95)

ABSTRACT

The notion of Ishikawa iterates is extended to a pair of multivalued maps and it is shown that under certain contractive conditions the limit of the sequence of Ishikawa iterates, when it converges, is a common fixed point of the maps.

Mathematics Subject Classifications (1991) : 54H25, 47H10

INTRODUCTION & PRELIMINARIES

Rhoades [11], Hicks and Kubicek [3] and several other mathematicians (cf. [9], [10], [12]) have shown that for any operator *T*, which satisfies certain contractive conditions, the limit of the sequence of Mann iterates (see [2] and [14]), whin it converges, is a fixed point of the operator *T*. On the other hand, Naimpally and Singh [9], under general contractive conditions, found that the limit of any convergent Ishikawa sequence (see [4]) is a fixed point of the operator; while Naidu and Prasad [8] studied the convergence of Ishikawa iterates of a pair of operators in respect of certain results given in [9]. Recently Kuhfittig [6] extended Krasnoselskii's method [5] to Mann iteration procedure for multivalued operators. Motivated by above ideas, Singh [15] defined Ishikawa iteration scheme for a multivalued operator and used it to approximate its fixed points.

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[#] The original article was not written in English. This, a page by page translation of the orignal article J. Natur. Phys. Sci. Vol. 4 (1-2) (1990), pages 187-198. is being published on the request of readers.

In the present paper, we use Ishikawa iteration scheme for a pair of multivalued operators to prove two theorems. The results obtained herein generalize the results of Singh [15]. Also, the results of Rhoades [11], Hicks and Kubicek [3], Naimpally and Singh [9], Naidu and Prasad [8] and Kuhfittig [6] can be obtained as special cases of our theorems.

Generally, we shall follow the notations used in Nadler [7] and Singh [15].

Let $(X, \|.\|)$ be a normed linear space, and let $CL(X)$ be the collection of nonempty closed subsets of X. Denote by $(CL(X), H)$ the Hausdorff metric space.

The following lemma is well known (see Rus [13] for details).

Lemma. Let $A, B \in CL(X)$, and let $q > 1$. Then for every $x \in A$, there exists $y \in B$ such that $d(x,y)\ D \leq qH(A,B)$.

For a nonempty closed convex subset C of X and for $S, T : C \to CL(C)$, as discussed in [15], we shall use the following iteration scheme.

(1) $x_0 \in C$,

(2) $y_n = b_n p_n + (1-b_n)x_n,\ n \geq 0,\ p_n \in Sx_n$,

(3) $x_{n+1} = (1-a_n)x_n + a_n q_n,\ n \geq 0$,

where $q_n \in Ty_n$ is such that

(4) $\|p_n - q_n\| \leq k^{-h} H(Sx_n, Ty_n),\ h, k \in (0, 1)$,

(5) $0 \leq a_n, b_n \leq 1$ for all n,

(6) $\lim_n a_n > 0$.

Note that if $T : C \to C$, then condition (4) is easily satisfied as $k^{-h} > 1$.

RESULTS

Theorem 1. Let C be a nonempty closed convex subset of a normed linear space X and let $S, T : C \to CL(C)$. Suppose that the sequence $\{x_n\}$, as defined earlier, converges to a point z. If for all $x, y \in C$

(7) $H(Sx, Ty) \leq k \max \{||x-y||.\ D(x, Sx), D(y, Ty), [D(y,Sx) + D(x, Ty)]\},$

where $k \in (0, 1)$, then $z \in Sz \cap Tz$

Proof : Since $x_n \to z$ and $\{a_n\}$ is bounded away from zero,

$$||x_{n+1} - x_n|| = a_n\,||x_n - q_n||$$

and

$$||q_n - z|| \leq ||q_n - x_n|| + ||x_n - z||$$

imply that $||q_n - x_n||$ and $||q_n - z||$ converge to zero. Further, from (2) and triangle inequality, we have

$$\begin{aligned}
||x_n - y_n|| &= b_n\,||x_n - p_n|| \leq ||x_n - p_n||,\\
||x_n - p_n|| &\leq ||x_n - q_n|| + ||q_n - p_n||,\\
||y_n - q_n|| &\leq ||y_n - x_n|| + ||x_n - q_n||\\
&\leq ||x_n - p_n|| + ||x_n - q_n||\\
&\leq ||x_n - q_n|| + ||p_n - q_n|| + ||x_n - q_n||\\
&\leq 2\,||x_n - q_n|| + ||p_n - q_n||
\end{aligned}$$

and

$$||y_n - p_n|| = (1 - b_n)\,||x_n - p_n|| \leq ||x_n - p_n||.$$

Therefore

$$||p_n - q_n|| \leq k^{-h} H(Sx_n, Ty_n)$$

$$\leq k^{-h} k \max\{||x_n - y_n||, D(x_n, Sx_n), D(y_n, Ty_n) ; [D(x_n, Ty_n) + D(y_n, Sx_n)]\}$$

$$\leq \quad k' \max\{\|x_n - p_n\|, \|y_n - q_n\|, \|x_n - q_n\| + \|y_n - p_n\|\}$$

where $k' = k^{1-h}$.

Hence

$$\|p_n - q_n\| \quad \leq \quad (2k'/(1 - k'))\|x_n - q_n\|.$$

Therefore making $n \to \infty$,

$$\|p_n - q_n\| \to 0, \qquad \|x_n - p_n\| \to 0,$$

and

$$\|p_n - z\| \to 0, \qquad \|y_n - q_n\| \to 0$$

and from (2)

$$\|y_n - z\| \to 0.$$

From condition (7),

$$\begin{aligned} D(z, Sz) \;&\leq\; \|z - q_n\| + D(q_n, Sz) \\ &\leq\; \|z - q_n\| + H(Ty_n, Sz) \\ &\leq\; \|z - q_n + k \max\{\|y_n - z\|, D(y_n, Ty_n), D(z, Sz), \\ &\qquad [D(y_n, Sz) + D(z, Ty_n)]\} \\ &\leq\; \|z - q_n\| + k \max\{\|y_n - z\|, \|y_n - q_n\|, D(z, Sz), \\ &\qquad \|y_n - z\| + D(z, Sz) + \|z - q_n\|\}. \end{aligned}$$

Hence by making $n \to \infty$, we have $z \in Tz$.

Now for $S, T : C \to CL(C)$, let us consider the following conditions.

(8a) $H(x, Sx) + H(y, Ty) \le a||x-y||, 1 \le a < 2,$

(8b) $H(x, Sx) + H(y, Ty) \le b[D(x,Ty) + D(y, Sx) + ||x-y||], 1/2 \le b < 2/3,$

(8c) $H(x, Sx)+H(y,Ty)+H(Sx, Ty) \le c[D(x,Ty)+D(y, Sx), 1 \le c < 3/2,$

(8d) $H(Sx, Ty) \le k \max \{||x-y||, Dx, Sx), D(y,Ty),$

$1/2 [D(x, Ty)+D(y,Sx)], 0 \le k < 1.$

Theorem 2. Let C be a nonempty closed convex subset of a normed linear space X, and let $\{x_n\}$ be the sequence as defined earlier converging to a point z. If for every $x, y \in C$, S and T satisfy at least one of the conditions (8a) - (8d) and,

(8e) $k^{-h} \le 2$ or $\lim_n b_n > (c-1-k^h)/c+1),$

then $z \in Sz \cap Tz.$

Proof. We shall repeatedly use the following statement due to Ciric [1]. For any $x \in C$ and $y \in B$, where $B \in CL(C)$, we have $||x-y|| \le H(x, B).$

Since $p_n \in Sx_n$ and $q_n \in Ty_n$, we have

$$2||p_n - q_n|| \le ||x_n - p_n|| + ||y_n - q_n|| + ||x_n - q_n|| + ||y_n - p_n||$$

$$\le H(x_n, Sx_n) + H(y_n, Ty_n) + ||x_n - q_n|| + ||y_n - p_n||.$$

Now, as in the case of Theorem 1, it can be easily varified that

$$\|x_n - q_n\| \to O_x \|q_n - z\| \to 0.$$

Setting $x = x_n$ and $y = y_n$ in (8a), we get

$$\begin{aligned} 2\|p_n - q_n\| &\leq a\|x_n - y_n\| + \|x_n - q_n\| + \|y_n - p_n\| \\ &\leq a\, b_n (\|x_n - q_n\| + \|q_n - p_n\|) + \|x_n - q_n\| \\ &\quad + (1 - b_n)(\|x_n - q_n\| + \|q_n - p_n\|). \end{aligned}$$

Hence

$$\|p_n - q_n\| \leq t_1 \|x_n - q_n\|,$$

where $t_1 = [2+(a-1)b_n]/[1-(a-1)b_n]$.

Again setting $x = x_n$ and $y = y_n$ in (8b), we have

$$\begin{aligned} 2\|p_n - q_n\| &\leq b(D(x_n, Ty_n) + D(y_n, Sx_n)) + \|x_n - y_n\| + \\ &\quad (\|x_n - q_n\| + \|y_n - p_n\|) \\ &\leq (b + 1)(\|x_n - q_n\| + \|y_n - p_n\|) + b\|x_n - y_n\| \end{aligned}$$

that is

$$\|p_n - q_n\| \leq t_2 \|x_n - q_n\|,$$

where $t_2 = (2 + 2b - b_n)/(1 - b + b_n)$.

If $x = x_n$ and $y = y_n$ in (8c), then

$$||p_n - q_n|| \leq t\, H(Sx_n, Ty_n), \text{ where } t = k^{-h}.$$

Therefore

$$(2t + 1)||p_n - q_n|| = t||p_n - q_n|| + t||p_n - q_n^{\bullet}|| + ||p_n - q_n||$$

$$\leq t(||p_n - y_n|| + ||y_n - q_n||) + t(||x_n - p_n|| + ||x_n - q_n||)$$

$$+ t\, H(Sx_n, Ty_n)$$

$$\leq t\, H(y_n, Ty_n) + t\, H(x_n, Sx_n) + t\, H(Sx_n, Ty_n)$$

$$+ t(||x_n - y_n|| + ||y_n - p_n||)$$

$$\leq ct\, D(x_n - Ty_n) + D(y_n, Sx_n) + t(||x_n - q_n|| + ||y_n - p_n||)$$

$$\leq t(c + 1)(||x_n - q_n|| + ||y_n - p_n||)$$

Thus $||p_n - q_n|| \leq t_3||x_n - q_n||$,

where $t_3 = [t(c + 1)(2 - b_n)]/[1 + t(1 - c(1 - b_n) + b_n]$

Finally, if we set $x = x_n$ and $y = y_n$ in (8d), then

$$||p_n - q_n|| \leq tk \max \{||x_n - y_n||, D(x_n, Sx_n), D(y_n, Ty_n),$$

$$1/2[D(x_n, Ty_n) + D(y_n, Sx_n)\}$$

$$\leq tk(2||x_n - q_n|| + ||q_n - p_n||).$$

Therefore

$$||p_n - q_n|| \leq t_4 \, ||x_n - q_n||,$$

where $t_4 = 2tk/(1-tk)$.

Hence for $x = x_n$ and $y = y_n$ we have

(9) $$||p_n - q_n|| \leq \max \{t_1, t_2, t_3, t_4\} \, ||x_n - q_n||.$$

Clearly, for any $b_n \in [\, 0\, ,\, 1\,]$, $t_4 > 0$ and $t_1, t_2 > 0$. Now keeping in view t_3 for any $b_n \in [\, 0\, ,\, 1\,]$, $t_3 > 0$ if $1 + t[1-c(1-b_n) + b_n] > 0$.

That is $1 + t(1-3(1 - b_n)/2 + b_n)$(since c and 3/2 may be sufficiently close).

Therefore $1 + t(1-3/2) > 0$ or $t < 2$.

It is clear that for any $b_n \in [\, 0\, ,\, 1\,]$, when $t = 2$, $t_3 > 0$. If $t > 2$ then $t_3 > 0$ provided

$$1 + t[1-c(1-b_n) + b_n] > 0, \text{ that is if } b_n > (c-1-1/t)/(c + 1).$$

Hence keeping in view (8e), $t_3 > 0$. Therefore from (9), it follows that

$$||p_n - q_n|| \to 0, \quad ||x_n - p_n|| \to 0,$$

and

$$||p_n - z|| \to 0 \text{ as } n \to \infty.$$

Now we shall show that z is a fixed point of S. Hence by setting

$x = z$ and $y = y_n$ in (8a), we have

$$\text{(10a)}\quad H(s\,,Sz) + H(y_n\,,Ty_n) \leq a||y_n - z||\,.$$

Similarly, from (8b),

$$||y - q_n|| + H(z, Sz) \leq b(D(y_n\,,Sz) + D(s, Ty_n) + ||y_n - z||$$

$$\leq b(||y_n - z|| + D(z, Sz) + ||z - q_n|| + ||y_n - z||)$$

$$\leq b(2||y_n - z|| + ||z - q_n|| + H(z, Sz)).$$

Therefore

$$\text{(10b)}\quad (1\text{-}b)H(z, Sz) \leq 2b(||y_n - z|| - (1 - b)||z - q_n||.$$

Again from (8c),

$$2H(z, Sz) \leq H(z, Sz) + ||z - y_n|| + H(y_n, Ty_n) + H(Ty_n, Sz)$$

$$\leq ||z - y_n|| + c(D(y_n,Sz) + D(s,Sz) + ||z - q_n||)$$

$$\leq (c + 1)\,||z - y_n|| + cH(z,Sz) + c||z - q_n||\,.$$

Thus

(10c) $(2-c)H(z,Sz) \leq (c+1)||z - y_n|| + c||z - q_n||.$

Finally, from (8d),

$$D(s,Sz) \leq ||z - q_n|| + H(Ty_n, Sz)$$

$$\leq ||z - q_n|| + k \max\{||y_n - z||, D(y_n, Ty_n), D(z,Sz),$$

$$1/2[D(y_n, Sz) + D(z, Ty_n)]\}.$$

Hence

(10d) $D(z, Sz) \leq ||z- q_n|| k \max\{||y_n - z||, ||y_n - q_n||, D(z,Sz),$

$$1/2[||y_n - z|| + D(z, Sz) + ||z - q_n||]\}.$$

Hence making $n \to \infty$ in (10a) - (10d), we have

$$H(z,Sz) = 0 \text{ and } D(z, Sz) = 0, \text{ and } z \in Sz.$$

Similarly, we can prove that $z \in Tz$.

Remark 1. If we set $S = T$ in theorems 1 and 2, we get results of Singh [15].

Remark 2. If we take $S, T : C \to C$ in Theorem 1, we get Theorem 1.2 of Naimpally and Singh [9].

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Journal of Natural & Physical Science Vol. 9-10 (1995-96) pp 25-42

ON FAN'S BEST APPROXIMATION AND FIXED POINT THEOREMS

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(Received 26-07-95)

ABSTRACT

In this paper an up-to date development of Ky Fan's best approximation theorem is given for single valued function as well as multivalued case. A few fixed point theorems are derived as corollaries.

Mathematics Subject Classifications : Primary 47 H10, Secondary 54 H25

Consider a function $F : C \rightarrow X$, where C is a nonempty subset of a normed linear space X. We seek a point $x \in C$ which is a best approximation for fx, i.e., the problem of existence of an $x \in C$ such that

$$||x - fx|| = d(fx, C) = inf\{||fx - y|| : y \in C\} \qquad (*)$$

We note that y is a solution of (*) if and only if y is a fixed point of $P_C of$, where P_C is the metric projection on C.

If f satisfies a suitable boundary condition, for example, $fx \in C$ for all $x \in C$, i.e. $f: C \rightarrow C$, then the set of solutions of (*) coincides with the fixed point set of f (see Park [11] for details).

In the study of fixed points we note that there are different theorems dealing with different boundary conditions. We give below a list of a few such conditions.

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1. Let $f: C \to X$ with $f(\partial C) \subseteq C$, ($\partial C$ stands for the boundary of C).
2. Let $f: C \to X$ and $x \neq fx$, then the line segment $[x, fx]$ has at least two points of C.
3. If $f: C \to X$ and $x \neq fx$ then there exists a number λ (real or complex depending on whether the vector space is real or complex) such that $|\lambda| < 1$ and $y = \lambda x + (1 - \lambda)fx \in C$.
4. If $f: C \to X$ and $x \neq fx$ for $x \in \partial C$ then there exists a $y \in C$ such that $||fx - y|| < ||x - fx||$.
5. If $f: C \to X$ and $x \neq fx$ for $x \in \partial C$ then

$$\lim_{h \to 0^+} \frac{d((1-h)x + hfx, C)}{h} < ||x - fx||.$$

In case C is a ball B of radius r and center 0, the boundary conditions are:

(i) If $f: B \to X$ and $fx = \alpha x$ for some $x \in \partial B$ then $\alpha \leq 1$

(Leray-Schauder condition).

(ii) If $f: B \to X$ then $||fx - x||^2 \geq ||fx||^2 - ||x||^2$ for all $x \in \partial B$

(Altman's Condition).

A similar set of boundary conditions, for dealing with the fixed point theorems of multifunctions, were considered.

The following well-known theorem due to Ky Fan[5] yields fixed point theorems under a set of different boundary conditions.

Theorem 1 : Let C be a nonempty compact, convex subset of a normed linear space X and $F: C \to X$ a continuous function. Then there is an $x_0 \in C$ such that

$$||x_0 - fx_0|| = d(fx_0, C) = inf\{||fx_0 - z|| : z \in C\}. \qquad (K)$$

(a) If additionally, $f(\partial C) \subseteq C$ is satisfied, then f has a fixed point.

(b) If additionally, for $x_0 \in \partial C$ with $x_0 \neq fx_0$ there exists a $y \in \overline{I_C(x_0)}$ such that $||x_0 - fx_0|| > ||y - fx_0||$, is satisfied then f has a fixed point.

(c) If additionally, for each $x_0 \in \partial C$ with $x_0 \neq fx_0$ the line segment $[x_0, fx_0]$ contains at least two points of C, is satisfied then f has a fixed point.

Proof. We consider case (c) and show that x_0 is a fixed point of f. By Theorem 1 there exists an $x_0 \in C$ such that

$$||x_0 - fx_0|| = d(fx_0, C).$$

Take $x_0 \neq fx_0$ and let $y = \lambda x_0 + (1 - \lambda) fx_0 \in C, \quad with \quad 0 \leq \lambda < 1$. Then

$$||x_0 - fx_0|| \leq ||y - fx_0|| = ||\lambda(x_0 - fx_0)|| < ||x_0 - fx_0||$$

since $\lambda < 1$, a contradiction. Hence $x_0 = fx_0$.

The compactness condition can not be dropped as is illustrated by the following example in a Hilbert space.

Let B be a unit ball in the Hilbert space l^2. Let f be defined on B by

$$f(x) = \{\sqrt{1 - ||x||^2}, x_1, x_2, \ldots, x_n \ldots\}.$$

Then $f: B \to B$ and $||fx|| = 1$. If (K) holds, then f must have a fixed point, i.e., $fx = x$ so $||x|| = 1$. This gives that

$$fx = x = \{0, x_1, x_2, \ldots, x_n, \ldots\} = \{x_1, x_2, x_3, \ldots, x_n, \ldots\},$$

i.e. $x_1 = 0 = x_2 = x_3 \ldots$, so $x = 0$, contradiction to $||x|| = 1$. So (K) does not hold true.

Now, results are given for approximatively compact sets. We need the following definitions (see Cheney [4]).

Let X be a normed linear space and C be a nonempty subset of X. Let $x \in X$. An element $y \in C$ is called an *element of best approximation* (or a *nearest point*) to x if

$$||x-y|| = d(x, C) = \inf \{||x-z|| : z \in C\}.$$

The set of best approximations to x is given by

$$P(x) = \{z \in C : ||x-z|| = d(x,C)\}.$$

The mapping $P : X \to C$ is called the *metric projection* of X onto C.

If $P(x) \neq \phi$ for every $x \in X$, then C is called *proximinal.* If $P(x)$ contains at most one element for every $x \in X$, then C is called a Chebyshev set.

For each x, $P(x)$ is a closed and bounded set, and is convex if C is convex. If C is a Chebyshev set, then P is a single-valued mapping of X onto C and is called *Chebyshev map* or *best approximation operator*. If the space is a Hilbert space, then it is called the *proximity map*. If C is a closed, convex subset of Hilbert space H, then C is a Chebyshev set and P is a nonexpansive map.

A sequence $\{y_n\}$ in C, where C is a subset of normed linear space X is called a *minimizing sequence* for $x \in X$ if $d(x,y_n)$ converges to $d(x,C)$.

A set C in X is said to be *approximatively compact* if for any $x \in X$ each *minimizing sequence* for x has a subsequence converging to an element of C.

If C is approximatively compact set then (i) C is proximinal, (ii) C is closed, (iii) Px is compact, (iv) if C is convex then Px is convex, (v) $P(A) = \cup\{Px : x \in A\}$ is compact for compact subset A in X.

A metric projection onto an approximatively compact set is upper semicontinuous (usc) (see Reich [15]).

A compact set is approximatively compact. However, the converse is not true. For example, the closed unit ball of an infinite dimensional uniformaly convex Banach space is approximatively compact but not compact.

The following is due to Reich [15] for approximatively compact sets.

Theorem 2: Let C be an approximatively compact, convex subset of a Banach space X and $f : C \to X$ continuous with $f(C)$ compact. Then there exists a $y \in C$ such that

$$||y - fy|| = d(fy, C).$$

Proof: Let Q be the metric projection on C. Define $F(x) = Q(f(x))$ for each $x \in C$.

Then F is usc and $F(x)$ is nonempty compact convex subset of C for each $x \in C$. Since $f(C)$ is relatively compact, so $F(C)$ is also relatively compact because the image of a compact set under an usc map with compact point images is compact. So the result follows from the Himmelberg [6].

Singh and Watson [19] proved the following result in Hilbert space for nonexpansive mappings.

Theorem 3: Let H be a Hilbert space and C is a closed, convex subset of H. Let $f : C \to H$ be a nonexpansive map with $f(C)$ bounded. Then there exists a $y \in C$ such that $||y - fy|| = d(fy, C)$.

Proof: Let $P : H \to C$ be the proximity map and $f : C \to H$. Then $P \circ f : C \to C$ is nonexpansive. Set $B = \overline{co(Pf(C))}$.

Then B is closed, bounded, and convex and $T = Pf : B \to B$. By Browder Fixed Point Theorem T has a fixed point; that is, $Pfx_0 = x_0$ for some $x_0 \in B$. Therefore, $||x_0 - fx_0|| = d(fx_0, C)$.

We give the following :

Corollary 4: If C is a closed, bounded, convex subset of H and $f : C \to H$. is a nonexpansive map then there is a $y_0 \in C$ such that

$$||y_0 - fy_0|| = d(fy_0, C).$$

The following fixed point theorems are derived.

1. Let B_r be a closed ball of radius r and centre 0 in Hilbert space H. Let $f: B_r \to H$ ba a nonexpansive map with the property that if $fx = \alpha x$ for some $x \in \partial B_r$ then $\alpha \leq 1$. Then f has a fixed point.
2. Let C be a closed convex subset of Hilbert space H and $f: C \to H$ a nonexpansive mapping. Let $f(C)$ be bounded and $f(\partial C) \subset C$. Then f has a fixed point [19].

Browder raised the following question.

Let C be a closed convex subset of Banach space X, and let $f: C \to C$ be a nonexpansive mapping . For any $k \in [0, 1)$ and any $x_0 \in C$, the mapping defined by

$$f_k(x) = kf(x) + (1 - k)x_0$$

maps C into itself and is a contraction with Lipschitz constant k. For k sufficiently close to 1, f_k is a contractive approximation of f.

By the Banach contraction principle there exists a unique fixed point x_k of f_k in C for any $k \in [0, 1)$; that is,

$$x_k = f_k x_k = kfx_k + (1-k)x_0 .$$

It is natural to ask if the sequence x_k converges to a fixed point of f. One cannot expect, in general, an affirmative answer to this question as there are nonexpansive mappings which do not have fixed points.

An affirmative answer in the following setting was given by Browder [1] and is stated next.

Theorem 5: Let C be a closed, bounded, convex subset of Hilbert space H and let $f: C \to H$ be a nonexpansive mapping. Define $f_k(x) = kf(x) + (1 - k)x_0$, where $0 < k < 1$ and x_0 is an arbitrary point in C. Let $f_k x_k = x_k$. Then x_k converges to y_0, where y_0 is a fixed point of f closest to x_0.

The following result was established by Singh and Watson [9] where f is not necessarily a self-map and C is not bounded.

Theorem 6: Let H be a real Hilbert space and C is a closed, convex subset of H. Let $f: C \to H$ be a nonexpansive mapping with $f(C)$ bounded and $f(\partial C) \subseteq C$. Suppose that $0 \in C$. Let $f_k(x) = kf(x) + (1 - k)x_0$ for some $x_0 \in C$ and $0 < k < 1$, $k \to 1$, and let $f_k x_k = x_k$. Then x_k converges strongly to y_0, where y_0 is the fixed point of f closest to x_0.

Proof: The fixed point set, $F(f)$, of f is nonempty [1] and $F(f)$ is closed and convex [1]. So there exists a unique closest point to x_0, say $y_0 = fy_0$. For the sake of convenience we take $x_0 = 0$. Now,

$$||x_k/k - y_0||^2 = ||fx_k - y_0||^2 = ||fx_k - fy_0||^2 \leq ||x_k - y_0||^2;$$

that is,

$$||x_k||^2 + k^2||y_0||^2 - 2k(x_k, y_0) \leq k^2(||x_k||^2 + ||y_0||^2 - 2(x_k, y_0)).$$

This gives

$$||x_k||^2 \leq \frac{2k}{1+k}(x_k, y_0) \leq (x_k, y_0).$$

So

$$||x_k|| \leq \left(\frac{x_k}{||x_k||}, y_0\right) \leq \left\|\frac{x_k}{||x_k||}\right\| ||y_0||,$$

that is $||x_k|| \leq ||y_0||$.

Since $\{x_k\}$ is bounded, x_{k_i} is a subsequence of $\{x_k\}$, converges weakly to x, that is $x_{k_i} = x_i \to x$, Then

$$\begin{aligned} ||x_i - fx_i|| &= ||k_i fx_i - fx_i|| \\ &= (k_i - 1)\, ||fx_i|| \to 0 \text{ as } i \to \infty. \end{aligned}$$

Hence $x_i - fx_i \to 0$.

Recall that a mapping f is demiclosed if $x_n \rightharpoonup x$ and $fx_n \to y$ and imply that $y = fx$ ($\rightharpoonup$ stands for weak convergence and $\to$ for strong convergence.)

Browder [1] proved that if $f : C \to H$ is a nonexpansive mapping, then $I - f$ is **demiclosed**. Since $I - f$ is demiclosed, we get $(I - f)x = 0$, that is, $fx = x$. Since $||x_i|| \leq ||y_0||$ so $||x|| \leq ||y_0||$. But y_0 is closest to $x_0 = 0$, so $||x|| = ||y_0||$. This implies that $x = y_0$, and so $x_i \rightharpoonup y_0$. Again

$$\begin{aligned} ||y_0||^2 \geq ||x_i||^2 &= ||x_i - y_0 + y_0||^2 \\ &= ||x_i - y_0||^2 + ||y_0||^2 + 2(x_i - y_0 , y_0). \end{aligned}$$

The last part of the right side goes to 0 as $i \to \infty$, giving us $||x_i - y_0|| \to 0$.

Thus x_i converges to y_0 strongly.

Since x_i is any subsequence of x_k, the sequence x_k converges strongly to y.

Remark: In this proof, the condition that $I - f$ is monotone, is not used. The following examples are given.

Example 1: Let $f : [-1 , 1] \to R$ be given by $f(x) = 1 - x$. Then $f(-1) = 2 \notin [-1 , 1]$, that is , f is not a self-map. But f does have a fixed point.

Example 2: Let $f : [0, \infty) \to R$ be given by $f(x) = 1/(1 + x)$. Then f is a nonexpansive mapping and $f(x_0) = x_0 = \left(\sqrt{5} - 1\right)/2$. But $[0, \infty)$ is not bounded.

Definition 7: Let $M \subseteq E$ be a convex subset. A mapping $g : M \to E$ is almost affine if and only if, for all $x, y \in M$ and $u \in E$, we have

$$||g(\lambda x + (1 - \lambda)y) - u|| \leq \lambda \, ||gx - u|| + (1 - \lambda)||gy - u||$$

for each λ, with $0 < \lambda < 1$.

Recently Prolla [14] proved the following:

Theorem 8: Let M be a nonempty, compact, convex subset of a normed linear space E and $g : M \to M$ a continuous, almost affine, onto mapping. Then, for each continuous mapping $f : M \to E$ there exists an $x \in M$ satisfying

$$||gx - fx|| = d(fx, M).$$

In case $g = I$, an identity function, we get Theorem 1.

Sehgal and Singh [18] proved this result under relaxed conditions, where an approximatively compact set is used in place of compact set C.

The following lemma is useful in the proof of the result.

Lemma 9: Let M be a non empty, approximatively compact convex subset of E and let $Q : E \to 2^M$ be the metric projection of E onto M defined by

$$Q(x) = \{y \in M : ||y - x|| = d(x, M)\}.$$

Then

(a) $Q(x) \neq \phi$ and

(b) $Q(A) = \cup\{Q(x) : x \in A\}$ is compact for any compact subset A in E.

The following result is given in [18].

Theorem 10: Let M be a nonempty, convex subset of a normed linear space E and let $Q : E \to 2^M$ be the metric projection satisfying

(a) $Q(x) \neq \phi$ *for each* $x \in E$, *and*

(b) Q maps compact subsets of E onto compact subsets of M.

Suppose $g : M \to M$ is continuous, almost affine, onto map satisfying for any compact set $D \subseteq M, g^{-1}(D)$ is compact.

Then for any continuous mapping $f : M \to E$ with $f(M)$ relatively compact, there exists an $x \in M$ such that

$$||gx - fx|| = d(fx, M).$$

Remarks 1:

1. It may be noted that if M is an approximatively compact set, then Q satisfies conditions (a) and (b) of Theorem 10.
2. If M is a compact subset of E, then clearly $F(M)$ is compact. Further, the continuity of g implies that for any compact set D, $g^{-1}(D)$ being closed is a compact subset of M. Thus Theorem 8 is a special case of Theorem 10.
3. In case $g = I$, an identity function, and M is compact, then Theorem 1 is obtained as a corollary.
4. In case $g = I$ and M is an approximatively compact subset of E, one gets Theorem 2.

The theorem dut to Carbone and Conti [2], that extends and unifies results in this area, is given below.

Theorem 11: Let X be a Banach space and C an approximatively compact, convex subset of X. Let $f : C \to X$ be a continuous map with $f(C)$ relatively compact. Let $g : C \to C$ be continuous, onto, proper map such that $g^{-1}(z)$ is an acyclic subset of C for every $z \in C$. Then there exists an $x \in C$ such that

$$||gx - fx|| = d(fx, C).$$

Recently, extending the result of Prolla [14], Carbone [3] proved the following.

Theorem 12: Let C be a nonempty convex subset of a normed linear space X, $f : C \to X$ a continuous map and $g : C \to C$ a continuous, onto and satisfy

$$||g(\lambda x_1 + (1-\lambda)x_2) - y|| \leq \max\{||gx_1 - y||, ||gx_2 - y||\}, \text{(almost quasiconvex)}.$$

Let D be a nonempty compact subset of C and B a nonempty subset contained in a compact convex subset of C such that for each $y \in C/D$, there exists an $x \in B$ such that $u = gx \in C$ and

$$||gy - y|| \geq || u - fy||.$$

Then there is a $z \in D$ such that

$$||gz - fz|| \leq ||y - fz|| \text{ for all } y \in C.$$

Further extension of this result is given below [12].

Recall that for $x \in C$ the inward set of C at x $I_C(x)$ is defined by $I_C(x) = \{y \in X : y = x + r(u - x),\ r > 0,\ u \in C\}$ and $\overline{I_C(x)}$ denotes the closure.

Theorem 13 : Let C be a nonempty convex subset of a normed linear space X, $f : C \to X$ a continuous map, and $g : C \to C$ a continuous, almost quasiconvex surjection. Let D be a nonempty compact subset of C and B a nonempty subset contained in a compact convex subset of C such that, for each $y \in C \backslash D$, there exists a point $x \in B$ such that $y = gx \in C$ and

$$||gy - fy|| > ||y - fu||.$$

Then there is a point $w \in D$ such that

$$|| gw - fw || \leq || z - fw ||$$

for all $z \in \overline{I_C(gw)}$.

More precisely, either

1. f and g have a coincidence point $w \in D$; or

2. there is a $w \in D$ such that $gw \in \delta C$ (boundary of C) and

$$0 < \| gw - fw \| \leq \| z - fw \| \text{ for all } z \in \overline{I_C(gw)}.$$

Now we discuss Ky Fan type results for multivalued mappings.

Let X and Y be normed linear spaces and 2^Y denote the family of nonempty subsets of Y. The mapping $F : X \rightarrow 2^Y$ is *upper semicontinuous* (*usc*) if and only if $F^{-1}(B) = \{x \in X : Fx \cap B \neq \phi\}$ is closed for each closed subset B of Y. Equivalently, F is *upper semicontinuous* (*usc*) if and only if for each $x \in X$ and open set U of Y with $Fx \subseteq U$ there exists a neighbourhood V of X with $Fz \subseteq U$ for each $z \in V \cap X$. F is lower semicontinuous (lsc) iff $F^{-1}(A) = \{x \in X : Fx \cap A \neq \phi\}$ is open for each open set A in Y. If F is both usc and lsc then F is said to be continuous.

We list some known fixed point theorems for multivalued mappings. The first is a result of Ky Fan's that extends the result of Kakutani to infinite dimensional space.

Theorem 14: Let C be a compact, convex subset of normed space X and $F : C \rightarrow 2^C$ an usc multifunction with Fx nonempty, closed, convex for all $x \in C$. Then F has a fixed point [5]

The following is due to Himmelberg [6].

Theorem 15: Let C be a nonempty convex subset of a normed linear space X and $F : C \rightarrow 2^C$ an usc multifucntion with Fx nonempty, closed, convex for all $x \in C$. *If $F(C)$* is contained in a compact subset of C then F has a fixed point.

Several interesting important results on fixed point theorems for multivalued mappings have been given by Reich [16]. He extended Ky Fan's result and then derives fixed point theorems. Reich [16] proved the following.

Theorem 16: Let C be a compact, convex subset of a locally convex space X and $F : C \to 2^X$ continuous, point-compact, point-convex map. Then there exists a $y \in C$ such that

$$d(y, Fy) = d(Fy, C). \qquad (K')$$

Recall that

$$d(A,B) = \inf \{||a - b|| : a \in A \text{ and } b \in B\}.$$

The following is due to Sehgal and Singh [17].

Theorem 17: Let E be a locally convex Hausdorff topological vector space and K a nonempty approximatively p-compact convex subset of E. If $F : K \to E$ is a continuous multifunction with nonempty closed convex values and $F(K)$ is relatively compact, then there exists an $x \in K$ such that

$$d_p (x, F(x)) = d_p (F(x), K).$$

The following simple example is due to Waters [21] and shows that even in the specisl case of uniformly convex Banach space E, continuity of F cannot be replaced by upper semicontinuity alone.

Example: Let $E = R^2$ with the euclidean norm and let $K = [0, 1] \times \{0\}$ Clearly K is a comvex and compact.

Define $F : K \to 2^E$ by

$$F(a,0) = \begin{cases} (0,1), \ if\, a \neq 0 \\ L = the \ \ line \ \ segment[(0,1), (1,0)], \text{if}\, a = 0 \end{cases}$$

F is a convex and compact valued multifunction. Then for any $A \subseteq E$,

$$F^{-1}(A) = \begin{cases} \phi, & \text{if } A \cap L = \phi \\ K, & \text{if } (0,1) \in A \\ (0,0), & \text{if } (0,1), \notin A,\ A \cap L \neq \phi. \end{cases}$$

Thus, F is an usc but not an lsc multifunction and FK is compact.

However, for any $(a, 0)$,

$$d((a,0), F(a,0)) > 1 = d(F(a,0), K) \text{ if } a \neq 0,$$

$$d((0,0), F(0,0)) = \frac{\sqrt{2}}{2} \neq d(F(0,0), K) = 0 \text{ if } a = 0.$$

Thus, the conclusion of Theorem 17 is not satisfied.

A convex space C is nonempty convex set in a vector space with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hulls are called polytopes [10]. Here X and Y are topologicas spaces.

A set $K \subset X$ is called σ-compact if K is the countable union of compact sets. A nonempty topological space is acyclic if all of its reduced Cech homology groups over rationals vanish.

For a given class $\mathbf{L}$ of multifunctions denote

$$\mathbf{L}(X,Y) = \{T : X \to 2^Y \mid T \in \mathbf{L}\};$$

$$\mathbf{L}_C = \{T = T_m T_{m-1}, \dots, T_1 | T_i \in \mathbf{L}\}.$$

Using the above notation, we have the following definitions.

(1) We say that F is a Kakutani map, and write $F \in \boldsymbol{K}(X,Y)$, if Y is a convex space and F is usc with nonempty, compact, convex values.

(2) F is an acyclic map, written $\mathbf{F} \in \mathbf{V}(X,Y)$ if F is usc with compact acyclic values.

(3) $F \in \mathbf{K}^+(X,Y)$ (resp. $\mathbf{V}^+(X,Y)$) if, for any σ-compact K for X there is a $\Gamma \in \mathbf{K}(K,Y)(\mathbf{V}(K,Y))$ such that $\Gamma x \subset Fx$ for each $x \in K$.

(4) $F \in \mathbf{K}^+(X,Y)$ (resp. $\mathbf{V}^+(X,Y)$) if, for any σ-compact K for X there is a $\Gamma \in \mathbf{K}_C(K,Y)(\mathbf{V}_C(K,Y))$ such that $\Gamma x \subset Fx$ for each $x \in K$.

It is known that $\mathbf{K}_C^+$ contains $\mathbf{K}$ and $\mathbf{K}_C$. Moreover, it is clear that $\mathbf{V}_C^+$ includes $\mathbf{V}_C$ and $\mathbf{K}_C^+$.

Lemma 18: Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E and $F \in \mathbf{V}_C^+(X,X)$. If F is a compact, then F has a ficed point [13].

Theorem 19: Let C be a nonempty approximatively compact, convex subset of locally convex Hausdroff topological vector space C, and suppose that $F \in \mathbf{V}_C^+(C, E)$ is a compact multifunction. Then for each continuous seminorm p on E there exists an $(x_0, y_0) \in F$ such that

$$p(x_0 - y_0) \leq p(x - y_0) \text{ for all } x \in C.$$

Proof: Consider the metric projection $Q_p : E \to 2^c$. Clearly, $Q_p(x)$ is nonempty, compact, and convex for every $x \in E$ and Q_p is an u.s.c. multifunction. Hence, $Q_p \in \mathbf{K}(E, C) \subset \mathbf{V}^+(E, C)$. Since it is clear that $\mathbf{V}^+$ is closed under composition, we have $Q_pF \in \mathbf{V}_C^+(C, C)$ and Q_pF is compact. Therefore, by Lemma 6.7, Q_pF has a fixed point. That is, there is a $y_0 \in Fx_0$ such that

$$x_0 \in Q_p y_0 = \{x \in C : p(y_0 - x) = d_p(y_0, C)\} \text{ (see [10, 11] for details).}$$

For a continuous multifunction $F : C \to 2^E$ Theorem 17 holds, where one cannot dispense with the lower semicontinuous. However, in Theorem 19, it is not needed that F be lower semicontinuous, only upper semicontinuity serves the purpose.

Example. Take $E = \mathbf{R}^2$ and $C = [0, 1] \times \{0\}$. Suppose $F \in \mathbf{V}(C, E)$ is given by

$$F(a,0) = F(a,0) \begin{cases} (0,1) \text{ if } a \neq 0. \\ [(0,1),(1,0), \text{ if } a = 0, \end{cases}$$

where $[A, B]$ stands for the closed line segment joining points A and B in the plane. Then F is not lower semicontinuous but $x_0 = (0,0)$ and $y_0 = (\frac{1}{2}, \frac{1}{2})$ satisfies the conclusion of Theorem 19.

Tecent work of Vetrivel, Veeramani and Bhattacharyya [20] is worth including. Here the multifunction F need not be convex-valued.

We need the following definition.

Definition 20: Let $F = F_1 o F_0$. The mapping $F : C \rightarrow E$ is said to have property (A) if and only if whenever there exists a $y \in F_0(x)$ for some $x \in C$ such that $d(y, C) = d(F_0x, C)$, then $d(F_1y, C) = d(F_0x, C)$.

They proved the following :

Theorem 21: Let C be a nonempty approximatively compact convex subset of locally convex Hausdorff topological vector space E. Let $F : C \rightarrow E$ be a multifunction satisfying property (A), where F_1 and F_0 are closed convex-valued continuous multifunctions and $F_0(C)$ is relatively compact. Then there exists an $x \in C$ such that

$$d(x, fx) = d(Fx, C).$$

In their proof, they used the Lassonde's fixed point theorem [7], whereas others used the Himmelberg's theorem [6].

A multifunction $F : X \rightarrow Y$, is denoted by $R(X, Y)$, if there exists two topological spaces X and Y such that (i) $F : X \rightarrow Y$ and (ii) $F = F_2 o F_1$, where F_2 and $F_1 \in R$, [$F \in R$ means $F : X \rightarrow Y$, F is usc and Fx is nonempty compact convex subset of Y (Y is a convex set in a topological vector space)].

The following is due to Lassonde [7].

Theorem 22: Let C be a nonempty convex subset of a locally convex Hausdorff topological vector space *X*. Then any compact multifunction $F \in R(C, C)$ has a fixed point.

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Journal of Natural & Physical Science Vol. 9-10 (1995-96) pp 43-52

FARTHEST POINTS IN NORMED LINEAR SPACES

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(Received 19-10-95)

INTRODUCTION

We recall that if G is a bounded subset of a (real or complex) normed linear space $Ex \in E$ and $g_0 \in G$, then g_0 is called a farthest point to x in G whenever

(1) $$\| g_0 - x \| = \sup_{g\in G} \| g - x \|$$

In section 2 of the present paper we shall give some characterizations of farthest poings to x in G. In section 3 we shall give some applications of farthest points in $C(Q)$ and $C_R(Q)$ where Q is compact.

CHARACTERIZATIONS OF FARTHEST POINTS.

$$\text{Let } F_G(x) = \left\{ g_0 \in G : \|g_0 - x\| = \| g_0 - x \| = \sup_{g\in G} \| g - x \| \right\}$$

In this section we shall consider some conditions that are related to farthest points from a bounded subset G of E.

Theorem 1: Let G be a bounded subset of a normed linear space E, $x \in E$ and $g_0 \in G$. Then $g_0 \in F_G(x)$ if and only if there exists an $f_0 \in E^*$ such that

(2) $$f_0 \in \mathcal{E}(S_{E^*})$$

(3) $$f_0(g_0 - x) = \sup_{g\in G} \| g - x \|$$

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To prove this theorem we need the following results.

Lemma 1 [2] : Let M be an extremal subset of a closed convex set A in a topological linear space L. Then

$$(4) \qquad \mathcal{E}(M) = \mathcal{E}(A) \cap M,$$

Where $\mathcal{E}(M)$ denotes the set of all extremal points of M.

Lemma 2 [7]: Let E be a normed linear space and let F be a nonempty convex subset $\{ x \in E : \| x \| = 1 \}$. Then the set

$(5) \qquad M_F = \bigcap_{x \in F} \{ f \in E^* : \| f \| = 1, f(x) = 1 \}$ is a nonempty extremal subset of the cell $S_E{}^* = \{ f \in E^* : \| f \| \leq 1 \}$ endowed with $\partial(E^*, E)$.

Corollary 1 [7] : Let E be a normed linear space and $x \in E, x \ddagger 0$. Then the set

$$(6) \qquad M_X = \{ f \in E^* : \| f \| = 1, f(x) = \| x \| \}$$

is a nonempty extremal subset of $S_E{}^*$ endowed with $\partial(E^*, E)$.

Proof of the Theorem (1): Let $g_0 \in F_G(x)$. Then, by the theorem 3.1 [3], there exists $f \in E^*$ such that $\| f \| = 1$.

$$f(g_0 - x) = \sup_{g \in G} \| g - x \|$$

By corollary 1, the set $M = \left\{ f \in E^* : \| f \| = 1, f(g_0 - x) = \sup_{g \in G} \| g - x \| \right\}$

is a nonempty extremal subset of $S_E{}^*$ endowed with $\partial(E^*, E)$. So by Krein-Milman theorem, the set of $\mathcal{E}(M)$ is a nonempty and hense, by Lemma 1, there exists an $f \in E^*$ such that (2) and (3) hold.

Conversely, assume that (2) and (3) hold. Then, by (2) and (3),

$$\| g_0 - x \| \geq \| g - x \| \qquad (g \in G)$$

whence $g_0 \in F_G(x)$.

Corollary 2: Let G be a bounded subset of normed linear space E, $x \in E$ and $g_0 \in G$. Then the following statements are equivalent :

(a) $g_0 \in F_G(x)$

(b) There exists $f \in E^*$ such that f satisfies (2).

$$|f(g_0 - x)| = \sup_{g \in G} ||g - x||, \tag{7}$$

$$|f(g_0 - x| \geq |f(g - x) \qquad (g \in G) \tag{8}$$

(c) There exists $f \in E^*$ such that f satisfies (2), (7) and

$$\operatorname{Re} f(g_0 - g)\overline{f(g_0 - x)} \geq 0 \tag{9}$$

Proof: Let $g_0 \in F_G(x)$. Then, by theorem 1, there exists $f \in E^*$ sych that (2) and (7) hold. by (7) and (2), we have

$$|f(g_0 - x)| \geq ||g - x|| \geq |f(g - x)| \qquad (g \in G)$$

Consequently, (a) $\Rightarrow$ (b). To prove (b) $\Rightarrow$ (c) assume that we have (b). Then by (8),

$$|f(g_0 - x)|^2 \geq |f(g - x)|^2 = |f(g - g_0)|^2 + f(g_0 - x)|^2$$

$$+ 2 \operatorname{Re} f(g - g_0) \overline{\overline{f(g_0 - x)}}$$

$$\geq |f(g_0 - x)^2 + 2 \operatorname{Re} f(g - g_0) \overline{\overline{f(g_0 - x)}}.$$

whence Re $f(g_0 - g) \overline{f(g_0 - x)} \geq 0$. $(c) \to (a)$ is trivial.

Example: Let $E = R^2$, the set

$G \{(x_1, x_2) : 0 \leq x_1 \leq 2; -1 \leq x_2 \leq 1\}$

and the point $x = (2.2)$. Then $(0, -1) \in f_G(x)$.

3. APPLICATIONS OF FARTHEST POINTS IN THE SPACES $C(Q)$ AND $C_R(Q)$

Theorem 3 : Let $E = C(Q)$ (Q compact), G a bounded subset of $x \in E$ and $g_0 \in G$ then $g_0 \in F_G(x)$ if and only if there exists a Radon measure μ (Real of Complex) on Q such that

$$(10) \quad |\mu|(Q) = 1$$

$$(11) \quad \frac{d\mu}{d|\mu|}(q) \in C(Q)$$

$$(12) \quad g_0(q) - x(q) = \left[\operatorname{sign} \frac{d\mu}{d|\mu|}(q)\right] \sup_{g\in G}\sup_{q\in Q} |g(q) - x(q)| \quad [q \in S(\mu)],$$

where (11) means that $\frac{d\mu}{d|\mu|}$ can be made continuous on Q by changing its values on a set of $|\mu|$ measure zero, in (12) $\frac{d\mu}{d|\mu|}$ is continuous function and $s(\mu)$ is the carrier of the measure μ.

Proof By theorem 3.1 [3] $g_0 \in F_G(x)$ if and only if there exists a Random measure μ on Q such that we have (10) and

$$(13) \quad \int_Q [g_0(q) - x(q)]\,d\mu(q) = \sup_{g\in G}\sup_{q\in Q} |g(q) - x(q)|$$

By (10), (13) and $g_0 - x \neq 0$, we show that

$$(14) \quad \frac{d\mu}{d|\mu|}(q) = \frac{\overline{g_0(q) - x(q)}}{\sup\limits_{g\in G}\sup\limits_{q\in Q} |g(q) - x(q)|} \; |\mu| - a.e \text{ on } Q$$

For that assume that there exists a set $A \subset Q$ with $\mu(A) > 0$ such that

$$(15)\quad \frac{d\mu}{d|\mu|}(q) \neq \frac{\overline{g_0(q)-x(q)}}{\sup\limits_{g\in G}\sup\limits_{q\in Q} |\, g(q)-x(q)\,|} \, |\, \mu\,| - a.e \text{ on } Q$$

then $Re\left[\frac{d\mu}{d|\mu|}(q)\{g_0(q)-x(q)\}\right] < \sup\limits_{g\in G} \| g-x \| \; |\mu| \text{ - } a.e. \text{ on } A.$

otherwise , by taking into account that

$$(16)\quad |\,\frac{d\mu}{d|\mu|}(q)\,| = 1 \;\; |\mu| \text{ -}a.e \text{ on } Q,$$

there would exist a set $A_1 \subset A$ with $\mu(A_1) \supset > 0$ such that

$$\sup_{g\in G} \| g-x \| < Re\left[\frac{d\mu}{d|\mu|}(q)\{g_0(q)-x(q)\}\right]$$

$$\leq |\frac{d\mu}{d|\mu|}(q)\;\{g_0(q) - x(q)\}|$$

$$\leq \| g - x \|$$

$$\leq \sup_{g\in G} \| g-x \| \; |\mu| \text{ - } a.e. \text{ on } A_1.$$

which implies that $\frac{d\mu}{d|\mu|}(q)\,[g_0(q) - x(q)]$ is real and positive $|\mu|$ -$a.e$ on A_1 and hence

$$\frac{d\mu}{d|\mu|}(q)[g_0(q) - x(q)] = \sup_{g\in G} \| g-x \| \; |\mu| \text{ - } a.e \;\text{ on } A_1.$$

$$\frac{d\mu}{d\,|\,\mu\,|}(q) = \frac{\sup\limits_{g\in G} \|\, g - x \,\|}{g_0(q) - x(q)} = \frac{\overline{g_0(q) - x(q)}}{\sup\limits_{g\in G} \|\, g - x \,\|} \,|\, \mu \,| - a.\text{eon } A_1$$

which contradicts (15) Consequently,

$$\text{Re} \int_Q [g_0(q) - x(q)]\, d\mu(q) = \text{Re} \int_Q [g_0(q) - x(q)] \frac{d\mu}{d|\mu|}(q)\, d\mu(q)$$

$$= \int_Q \text{Re} \frac{d\mu}{d|\mu|}(q) \{g_0(q) - x(q)\}\, d\mu(q)$$

$$< \int_Q \sup_{g\in G} \|g - x\|\, d\,|\,\mu\,|(q)$$

$$\text{i.e.} \qquad < \sup_{g\in G} \|\, g - x \,\|$$

which contradicts (13). Hence we obtain (14). By changing the values of $\frac{d\mu}{d|\mu|}$ on a set of $|\mu|$ measure zero such that (14) holds everywhere on Q, we obtain (11). By (16).

$$(17) \qquad |\frac{d\mu}{d|\mu|}| = 1 \qquad [q \in S(\mu)]$$

Indeed, if there exists a $q_0 \in S(\mu)$ such that $|\frac{d\mu}{d|\mu|}(q) \neq 1$,

then, by taking an open neighbourhood U of q_0 such that

$|\frac{d\mu}{d|\mu|}(q)| \neq 1$ $(q \in U)$, we obtain, since $q_0 \in S(\mu)$ $|\mu|$ $(U) > 0$ which is a contradiction to (16). Thus, by (14) (everywhere on Q) we obtain (12).

Conversely, let the conditions (10) - (12) be satisfied. Then, by (12), (11) and (10),

$$\int_Q [g_0(q) - x(q)]d\mu(q) = \int_Q \sup_{g\in G} \max_{q\in Q} [g(q) - x(q)] \; sign \frac{d\mu}{d|\mu|}(q) d\mu(q)$$

$$= \sup_{g\in G} ||g - x|| \int_Q Sign \frac{d\mu}{d|\mu|}(q) \; \frac{d\mu}{d|\mu|}(q) d|\mu|(q)$$

$$= \sup_{g\in G} ||g - x|| \int_Q |\frac{d\mu}{d|\mu|}(q)| \; d|\mu|(q)$$

$$= \sup_{g\in G} ||g - x||,$$

which complets the proof.

For a compact space Q, we denote by $C_R(Q)$ the space of all continuous real valued function on Q, endowed with the ususal vector operations and with norm $||x|| = \max_{q\in Q} |x|(q)$.

Theorem 4 : Let $E = C_R(q)$ *(Q compact)*, G a bounded subset of E, $x \in E$ and $g_0 \in G$. Then $g_0 \in F_G(x)$ if and only if there exist two disjoint sets $Y^+_{g_0}$ *and* $Y^-_{g_0}$ closed in Q and a Radon measure μ on Q such that

(18) $|\mu|(Q) = 1$,

μ is nondecreasing on $Y^+_{g_0}$ $noninvrea \sin g \; on \; Y^-_{g_0}$ and

(19) $S(\mu) \subset Y^+_{g_0} \cup Y^-_{g_0}$ [$S(\mu)$ - the carrier of μ]

$$(20)\quad g_0(q)-x(q) = \begin{cases} \sup\limits_{g\in G}\max\limits_{q\in Q}|g(q)-x(q)|, q\in Y^+_{g_0} \\ -\sup\limits_{g\in G}\max\limits_{q\in Q}|g(q)-x(q)|, q\in Y^-_{g_0} \end{cases}$$

Proof : By theorem 3.1 [3], $g \in F_G(x)$ if and only if there exists a Radon measure μ on Q such that (10) and (13) hold. We shall show that these conditions are equivalent to (18) - (20). Assume that we have (10) and (13). Now define

$$(21)\quad Y^+_{g_0} = \left\{ q \in Q : g_0(q) - x(q) = \sup_{g\in G}\max_{q\in Q}|g(q)-x(q)| \right\}$$

$$(22)\quad Y^- = \left\{ q \in Q : g_0(q) - x(q) = -\sup_{g\in G}\sup_{q\in Q}|g(q)-x(q)| \right\}$$

Then, by definition, $Y^+_{g_0}$ and $Y^-_{g_0}$ are disjoint closed sets in Q and so we obtain (20). To prove (19) Let μ be decreasing on $Y^+_{g_0}$. Then there would exist a set $A \subset Y^+_{g_0}$ with $\mu(A) > 0$ such that

$\mu(A) < |\mu|(A)$. So

$$\int_A |g_0(q)-x(q)|d\mu(q) = \int_A \sup_{g\in G}\max_{q\in Q}|g_0(q)-x(q)|d\mu(q)$$

$$= \sup_{g\in G} ||g-x||\mu(A)$$

$$< |\mu|(A) \sup_{g\in G} ||g-x||$$

whence by (18)

$$\int_Q [g_0(q) - x(q)]d\mu(q) < \int_Q \sup_{g \in Q} ||g - x|| d|\mu|(q) = \sup_{g \in G} ||g - x||,$$

which contradicts (13). Hence μ is nondecreasing on $Y^+_{g_0}$. Similarly it can be shown that μ is nonincreasing on $Y^-_{g_0}$.

If there exists a $q_0 \in S(\mu)$ such that $Q \notin Y^+_{g_0} \cup Y^-_{g_0}$, then

$$|g_0(q_0) - x(q_0) < \sup_{g \in G} ||g - x||, .$$

by taking a neibourhood U of q_0 such that

$$|g_0(q)\text{-}x(q) \ | < \sup_{g \in G} ||g - x||, \qquad (g \in U),$$

we would have $|\mu| (q) > 0$, since $q_0 \in S(\mu)$. So

$$\int_U [g_0(q) - x(q)]d\mu(q) < \int_U |g_0(q) - x(q)| \ d|\mu|(q)$$

$$\leq \int_U \sup_{g \in G} ||g - x|| \, d|\mu|(q)$$

whence, by (10),

$$\int_Q [g_0(q) - x(q)]d\mu(q) < \int_Q \sup_{g \in G} ||g - x|| d|\mu|(q) = \sup_{g \in G} ||g - x||.$$

which contradicts (13). Hence $S(\mu) \subset Y^+_{g_0} \cup Y^-_{g_0}$.

Conversely, assume that there exists two disjoint closed sets $Y^+_{g_0}$ and $Y^-_{g_0}$ in Q and a randon measure μ on Q such that (18) - (20) hold. Then by (19), (20) and (18).

$$\int_Q [g_0(q)-x(q)]d\mu(q) \le \int_{s(\mu)\cap Y^{+}_{g_0}} \sup_{g\in G}\ \max_{q\in Q}|g(q)-x(q)|\, d\mu(q)$$

$$+\int_{s(\mu)\cap Y_{g_0}} -\sup_{g\in G}\ \max_{q\in Q}|g(q)-x(q)|\, d\mu(q)$$

$$\int_0 -\sup_{g\in G}\ \max_{q\in Q}|g(q)-x(q)|\, d\mu(q) = \sup_{g\in G}\|g-x\|$$

Thus gives (13). Thus (18) - (20) are none other then (10) - (13).

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Journal of Natural & Physical Sciences Vol. 9-10 (1995-96) pp 53-60

μ AND λ INVARIANTS OF A p-ADIC MEASURE

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(Received 14-04-95)

ABSTRACT

The Iwasawa Invariants of p-adic measure (a power series $F(x)$) will be characterized by Kummer-type congruences. As an application, a new generalization of the Kummer congruence is obtained. This is expressed in new compact notation.

Mathematics subject classifications (1991) 11B68, 11S40

Keywords : Kummer congruences, Iwasasa μ and λ Invariants, Bernoulli numbers, p-adic measure.

INTRODUCTION

Let p be a (fixed) prime number and put $q = 4$ for $p = 2$ and $q = p$ for $p > 2$. Let $\mathfrak{O}$ be the ring of intigers of a finite extention K of Q_p, with a local parameter π, $[k:Q_p] = e$. Denote by ord_π the π-adic valuation on K (i.e., $ord_\pi(x) = t$ means $x \in \pi^t\mathfrak{O}/\pi^{t+1}\mathfrak{O}$). Iwasawa invariants, $\mu(F)$ and $\lambda(F)$, of a power series $F(x) = \sum_{i=0}^{\infty} a_i x^i \in \mathfrak{O}[[x]]$ are defined by,

$$\text{(1.1)} \quad \mu(f) = min\{ord_\pi(a_i) : i \geq 0\} \text{ and } \lambda(f) = min\{i : \mathrm{ord}_\pi(a_i) = \mu(f)\}.$$

In his proof of the Ferrero - Washington theorem ($\mu = 0$), Sinnot [9] showed that the μ-invariants of p-adic measure and its Γ-transform are equal. In a related paper [7], Kida proved a theorem concerned with the λ-invariants of a p-adic measure supported on $1 + pZ_p$. Metsänkylä [8] gave a relation between Kummer type congruences and Iwasawa invariants provided that $\lambda < p$ and $\mu = 0$. In this paper Iwasawa invariants will be characterized by Kummer-type congruences and a new generalization of Kummer congruence is proven.

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2. The research upon which this paper is based was started at the University of Illinois at Urbana. I should like to express my appreciation to the staff of the math department there for their help and encouragement.

Define $G_c(j,F,n) \in K$ for $j,n \geq 0$ by

$$(1.2) \quad G_c(j,F,n) = \left(q_j^{-1}\Delta_c\right)F(\beta\alpha^n - 1),$$

where $\alpha \in qZ_p$ and $\beta \in 1 + \pi\mathfrak{O}$ are constants, $c > 0$ is an integer, $\left[q_J^{-1}\Delta_c\right]$ is the binomial coefficient operator acting on $n\left(\binom{x}{j} = \frac{x(x-1)\dots(x-j+1)}{j!}\right)$ and difference operator Δ_c is defined by $\Delta_c x_n = x_{n+c} - x_n$.

Remark : It is clear that $G_c(j,F,n)$ depends on λ, β and the infinite sum, $f(\beta\lambda^n - 1)$,converges in the π-adic topology because $\beta\alpha^n - 1 \in \pi O$.

(1.3) Theorem : (i) $G_c(j,F,n) \in \mathfrak{O}$ and $G_c(j,F,n) \bmod \pi\mathfrak{O}$ is independent of n.

(ii) if $(c,p) = 1$ and $\alpha \notin 1 + pqZ_p$, then

$\mu(F)=min\{ord_\pi(G_c(j,F,n)) : j \geq 0\}$ and $\lambda(F)=min\{j : ord_\pi(G_c(j,F,n)=\mu(F)\}$.

PROOF OF THE THEROEM:

Let $s(l, m)$ and $S(r, h)$ be the Stirling numbers of first and second kinds respectively. The binomial coefficient operator, $\left[q_J^{-1}\Delta_c\right]$, is defined by

$$(2.1) \quad \left(q_J^{-1}\Delta_c\right)\tau^n = (1/j!)\left(\sum_{i-0}^{j} s(j,i)q^{-i}D_c^i t^n\right),$$

where $\Delta_c^i(\tau^n) = \sum_{i-0}^{j}\binom{i}{k}(-1)^{i-k_\tau n+kc} = (\tau^c - 1)^i\tau^n$ Hence τ^n is an eigenvector of $\left(q_j^{-1}D_c\right)$ with eigenvalue $\begin{pmatrix}\eta(\tau)\\ j\end{pmatrix}$ where $\eta(\tau) = (\tau^c - 1)/q$

Define $A(t,k,n) \in K$ by

(2.2) $$A(t,k,n) = \left(q_t^{-1}D_c\right)(\beta\alpha^n - 1)^k \quad \text{for } t,\ k,\ n \geq 0.$$

For brevity we write $mod\ p^r$ $(mod\ \pi)$ for $mod\ p^rZ_p$ $(mod\ \pi\mathfrak{O})$

(2.3) **Lemma:** (i) $ord_\pi A(t,k,n) \geq max\ \{0,\ k\text{-}2et\text{-}ord_\pi)\ t!)\}$.

(ii) $A(t,k,n)$ $mod\ \pi$ is independent of n.

(*We write* $A(t,k) \not\equiv A(t,k,n)\ mod\ \pi$.)

(iii) $A(t,k) \not\equiv 0\ (mod\ \pi)$ if $k > t$.

(iv) $A(k,k) \not\equiv r^k c^k\ (mod\ \pi)$, where $\alpha = 1 = \alpha q$.

In particular (a) $A\ (t,k,n) \to 0$ as $k \to \infty$ and

(b) $A(k,k) \not\equiv 0 (mod\ \pi)$ if $\alpha \notin 1 + pqZ_p$ and $(c,p) = 1$.

Proof: (i) From the definition $ord_\pi A(t,k,n) \geq k\text{-}2et\text{-}ord_\pi).t\ !)\}$.

Since $\alpha \in 1 + pZ_p$, $\eta(\alpha^i) \notin Z_p$ and hence $\begin{pmatrix}\eta(\alpha^i)\\ t\end{pmatrix} \in Z_p$. We get from $(\beta\alpha^n\text{-}1)^k = \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}(\beta\alpha^n)^i$ that

$$A(t,k,n) \overset{\cdot}{=} \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}(\beta\alpha^n)^i\begin{pmatrix}\eta(\alpha^i)\\ t\end{pmatrix} \in \mathfrak{O}\ \square.$$

(ii) We have $A(t,k,n) \not\equiv \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\binom{n(\alpha^{i})}{t}(\bmod \pi)$ (because $(\beta\alpha^{n} \in 1 + \pi\,\mathfrak{O})$ from which it follows that $A(t,k,n) \bmod \pi$ is independent of n□.

(iii) and (iv) Recall that the p-adic exponential function, $exp_p(x)$, (we write e^x for simplicity) converges for all $x \in qZ_p$. Hence for all $x \in Z_p$,

$$(e^{qX} - 1)^{h} = (qx)^{h}\,(1+ \sum_{r=1}^{\infty} qb_r x^{r}) \text{ with } b_r \;\varepsilon\; Z_p .$$

(Because $e^{qX} - 1 = qx(1+\sum_{r=1}^{\infty} \frac{(qx)^r}{(r+1)!})$,$ord_2\,(4^r/(r+1)!) \geq 2r - r > 0$ and if $p \neq 2$, $ord_p\,(p^r/(r+1)!) \geq r - (r/(p-1)) > 0$.) Since e^q is a topological generator for $1 + qZ_p$ there is a $d \in Z_p$ such that $\alpha = e^{qd}$ and $d \not\equiv \gamma\,(mod\, p)$. Hence

$$\eta(\alpha^{i})^{h} = (e^{qdci} - 1)^{h}/q^{h} = (dci)^{h}\,(1+ \sum_{r=1}^{\infty} qb_r\,(dci)^{r}).$$

$$\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\,\eta(\alpha^{i})^{h} = \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\,(dci)^{h}\,(1+ \sum_{r=1}^{\infty} qb_r\,(dci)^{r})$$

$$= k!\,d^{h}c^{h}\,(S(h,k) + \sum_{r=1}^{\infty} S(r+h,\,k)\,qb_r\,d^{r}c^{r}).$$

(Because $\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\,i^{m} = k!\,S(m,\,k)$.) Interchanging summation we get,

$$(2.4)\; A(t,\,k) \not\equiv \sum_{k=0}^{t}\frac{k!}{t!}\,s(t,h)\,d^{h}c^{h}\;S(h,\,k) + \sum_{r=1}^{\infty}\; S(r+h,\,k)\,qb_r\,d^{r}c^{r})(mod\,\pi).$$

Hence $A(t,\,k) \not\equiv \sum_{k=0}^{t}\frac{k!}{t!}\,s(t,h)\,d^{h}c^{h}\;S(h,\,k) \not\equiv \delta_{t,k}\,\gamma^{k}c^{k}\,(mod\,\pi)$ for all $k \geq t$.

(Because $s(m, j) = 0 = S(m, j)$ whenever $0 \leq j < m$ and $d \not\equiv \gamma\,(mod\, p)$ ■

Remark: Suppose $\gamma \not\equiv 1\ (mod\, p)$ and $c \not\equiv 1\ (mod\, p)$. if $k < t < p$ then $A(t,k) \not\equiv 0\ (mod\ \pi)$.

Proof of the theorem: From definitions 1.2 and 2.2, one gets $G_c(j,F,n) = \left(q_j^{-1}\Delta_c\right) \sum_{i=0}^{\infty} a_i(\beta\alpha^n - 1)^i = \sum_{i=0}^{\infty} a_i A(j, i, n)$ and then apply 2.3 to get,

$$(2.5)\quad G_c(j,F,n) = \sum_{i=0}^{j} a_i A(j, i, n)\ (mod\ \pi).$$

Then 1.3(i) follows from (i) and (ii) of 2.3.

We can assume that $\mu(F)=0$ (Because, if $\mu(F) > 0$, replace $F(x)$ by $E(x) = \pi^{-\mu(F)} F(x)$, then $\mu(E) = 0$ and $\pi^{\mu(F)} G_c(j, E, n) = \pi^{\mu(F)} \left(q_j^{-1}\Delta_c\right) E(\beta\alpha^n - 1) = \left(q_j^{-1}\Delta_c\right) F(\beta\alpha^n - 1) = G_c(j, E, n)$). But $\mu(F) = 0$ means $a_i = 0\ (mod\ \pi)$ for all $i < \lambda(F)$ and $a_{\lambda(F)} \not\equiv 0\ (mod\ \pi)$. Hence, one gets by using 2.5 and 2.3 that $G_c(j, F, n) \not\equiv 0\ (mod\ \pi)$ for all $j < \lambda(F)$ and $G_c(\lambda(F), F, n) \not\equiv a_{\lambda(F)} A(\lambda(F), \lambda(F), n) \not\equiv 0\ (mod\ \pi)$, from which it follows that $min\{ord_\pi(G_c(j,F,n)) : j \geq 0\} = 0 = \mu(F)$ and $\lambda(F) = min\{j : ord_\pi(G_c(j,F,n) = 0\}$ ■

CONSEQUENCES AND APPLICATIONS

Let χ be a non-trivial even Dirichlet character with conductor rp^t, $t \geq 0$, $(r,p) = 1$ and $K = Q_p(\chi)$. If $r = 1$ assume that $\chi(m) \neq \chi(m+p^k)$ for all $k \geq 0$. From Iwasawa theory [4-5] there is a power series $F(x, \chi) \in \mathfrak{O}[[x]]$ associated to χ such that $F(\zeta(1+rp)^s, \chi) = L_p(s, \chi)$, where $L_p(s, \chi)$ is a p-adic L-function associated to χ and $\zeta = \chi(1 + rp)^{-1}$. Set $\alpha = (1 + rp)^{-1}$ and $\beta = \zeta(1 + rp)$ and write $G(j, \chi, \eta)$ for $G(j, F, n)$. Then for all $n \geq 1$,

$$G_c(j, \chi, n) = \left(q\,{}_j^{-1}\Delta_c\right) F(\zeta(1+rp)^{1-n}, \chi, = \left(q\,{}_j^{-1}\Delta_c\right) L_p(1-n, \chi) = \left(q\,{}_j^{-1}\Delta_c\right) \beta_{\chi,n},$$

where $\beta_{\chi,n} = L_p(1 - n, \chi)$ (because $L_p(1 - n, \chi) = -\frac{1}{n}(1 - \chi\omega^{-n}(p)\,p^{n-1}) \times B_{,n,}\chi\omega^{-n}$, it represents an *extended generalized Bernoulli number*, where ω is the Teichmüller character). Let $K_c(j, \chi, n) = q^{-j}\nabla^j \beta_{n,\chi}$ then $K_c(j, \chi, n) = \sum_{i=0}^{j} i!\, G_c(i, \chi, n)\, S(j, i) \in \mathfrak{O}$. Hence we get Metsänkylä's Theorems 1 and 2 [8] as a consequence of 1.3.

(3.1) Theorem 1 & 2 [8]: Suppose that $F(x, \chi) = \sum_{i=0}^{j} a_i x^i$ and $0 \le h < p$.

Then following three conditions are equivalent.

(i) $a_i \not\equiv 0 \ (mod\ \pi)$ for all $0 \le i \le h$.

(ii) $K_1(i,\chi,n) \not\equiv 0 \ (mod\ \pi)$ for all $0 \le i \le h$.

(iii) $K_{p-1}(i,\chi,n) \not\equiv 0 \ (mod\ \pi)$ for all $0 \le i \le h$.

Note : The fact that theorem 1.3 is independent of c gives the equivalence of (ii) and (iii)

Since $K_c(j,\chi,n) \not\equiv \sum_{i=0}^{p-1} i!\, G_c(i, \chi, n)\, S(j, i) \ (mod\ \pi)$ and $S(p-1+j, i) \not\equiv S(j, i) \ (mod\ p)$ whenever $j > 0$, we get the following result.

(3.2) Theorem (Periodicity of Kummer Congruences)

If $j, j' > 0$ and $j \not\equiv j' \ (mod\ p-1)$ and n, n' arbitrary then

$$K_c(j, \chi, n) \not\equiv K_c(j, \chi, n) \ (mod\ \pi).$$

Since $\mu(\omega^m) = 0$ [1] one gets by applying Theorem 1.3 with $\chi = \omega^m \neq 1$.

(3.3) Theorem (Generalized Kummer Congruence):

(i) $G_c(j, \omega^m, n)\ \varepsilon\ Z_p$ and $G_c(j, \omega^m, n) \bmod p$ is independent of n.

(ii) If $p \nmid c$ and $2 \mid m$ then there exists $i \geq 0$ such that $G_c(i, \omega^m, n) \not\equiv 0 \pmod p$ and $\lambda(\omega^m) = \min \{ j : G_c(j, \omega^m, n) \not\equiv 0 \pmod p\}$.

One can deduce the classical Kummer congruence from $K_c(j, \chi, n) \in \mathfrak{O}$ as follows : Set $c = p - 1$ $(p > 2)$ and $\chi = \omega^n (\neq 1)$ then $p^j K_{p-1}(j, \omega^m, n) \in p^j Z_p$.

i.e. $$\Delta^j \beta_{n,\omega^n} = -\sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} \frac{(1-p^{n-1})}{n+k(p-1)} B_n \not\equiv 0 \ (\mathrm{mod}\ p^j Z_p).$$

Theorem 1.3 together with a result in [3] can be used to prove the Iwasawa criterion [6] for $\mu(\omega^m) = 0$. Lemma 2.3 is also important its own right. For example, one can use 2.3 to give a proof of Kida's theorem [7] on λ-invariants of Γ-transform of a p-adic measure [3].

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Journal of Natural & Physical Science Vol. 9-10 (1995-96) pp 61-71

AEROMYCOFLORA OVER POTATO FIELDS

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(Received 03-04-95 and after revision 10-8-96)

ABSTRACT

Aeromycology over potato fields at Kurukshetra was studied consecutively for two years in relation to environmental conditions using culture plate method. A total of 25 and 29 fungal forms were trapped from the air in 1987-88 and 1988-89 crop season. Cladosporium cladasporioides was the chief constituent contributing 59.33% and 57.93% of the total fungi in the two respective years. The other dominant fungi in the two successive years were those of Alternaria, alternata, Aspergillus, flavus, A. niger, Cladosporium, herbarum, Epicoccum, nigrum, Fusarium equisetum, Penicillium, citrinum, P., cyclopium and Trichothecium, roseum. In the diurnal cycle the fungi showed an evening tendency. Rain has profound effect on the aeromycoflora.

Key words : Aeromycoflora, Cladosporium cladosporioides, Potato.

INTRODUCTION

Aerobiological studies conducted in various parts of India revealed the richness of airspora [15]. Standing vegetation has a great influence

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on airspora of any place and it changes with changes in weather [2]. In the present study aeromycoflora of potato fields was studied consecutively for two years in relation to environmental conditions.

MATERIALS AND METHODS :

Aeromycology of potato fields was recorded during 1987-88 and 1988-89 crop season at Kurukshetra (Haryana) by culture plate method. Four petri dishes containing Martin's agar medium were exposed for five minutes at one and half feet height in the potato fields at 6.00 a.m., 12.00 noon, 6.00 p.m. and 12.00 night. These plates were incubated at 25°C and the number of fungal colonies appearing were counted and identified. Relative humidity and temperature were recorded by placing hygrographs and thermographs in the fields and rainfall was recorded from soil and water testing laboratory, Kurukshetra.

RESULTS:

1. **Components** : The spore content of air over the potato fields was rich both qualitatively and quantitatively. A list of fungal spore types and the percentage contribution to total aeromycoflora was given in Table 1. A total of 25 and 29 fungal forms trapped from the air in 1987-88 and 1988-89 crop season. Acremonium, implicatum, Alternaria, longipes, Bipolaris, hawaiiensis, Emericella nidulans and Epicoccum, purpurascens were restricted to first year. While Aspergillus, fumigatus, Fusarium, moniliforme, Myrothecium verrucaria, Nigrospora sphaerica, Penicillium purpurogenum and Periconia, saraswatipurensis were trapped only from the aeromycoflora of second year crop season. Of all the components Cladosporium cladosporioides was the chief constituent contributing 59.33% and 57.93% of the total fungi in the two respective years. The other abundant colonies in the two successive years were those of A. alternata, A. tenuissima, A. flavus, A., niger, C., herbarum, E.

nigrum, F., equiseti, P., citrinum, P., cyclopium, and T., roseum (Table 4 and 5)

2. **Diurnal and seasonal periodicities :** The fungal content of air over potato fields varied between 7.4 and 157.4 propagules per 100 cm^3 (Table 2) in the first year and between 6.5 and 117.8 propagules per 100 cm^3 in the second year (Table 3). In the diurnal cycle the fungi showed an evening tendency. On the basis of statistical analysis aeromycoflora varied significantly seasonally on one hand and diurnally it varied insignificantly on the other hand in 1987-88. While in 1988-89, aeromycoflora over potato varied insignificantly both seasonally and diurnally.

3. **Effect of weather :** Climatic changes profoundly influenced the aeromycoflora over potato fields. It appears from the data (Fig. 1) that high temperature and low relative humidity in the month of October and November favoured the aeromycoflora but comparatively low temperature and high humidity have adversely affected the aeromycoflora. Of all the climatic factors, rain is the single important factor that causes major fluctuations in the spore content of the air. The sampling taken after rains showed a decrease in the spore content in the air. In 1987-88 the first spell of rain i.e. 6.5mm was recorded on 14.12.87. It was found that it increased the spore load in the air which is evident from the third sampling taken on 25.12.87. The rain was frequent in the months of December 1988 and January 1989 with a prolonged spell of fog in the nights and mornings. This type of atmosphere decreased the spore load in the atmosphere and lowest counts were trapped in this period.

DISCUSSION:

In the diurnal cycle the aeromycoflora showed evening maxima and night and morning minima and this trend is in confirmity with earlier reports from India [1, 7, 12]. The fluctuations in the aeromycoflora can be attributed to prevailing environmental conditions. The maximum aeromycoflora on 25.12.87 indicates the effect of first rain in bringing more spores into the air was also observed by Sreeramulu and Ramalingam [14] and Malliah and Rao [8]. The least count on 21.01.88 can be attributed to the low temperature, foggy night and prolonged periods of dew on the surface of plants and other inanimate objects. In the next year the highest aeromycoflora was recorded on 30.01.88. This is due to absence of rain with optimum temperature and humidity in the atmosphere. The maximum aerospora in the dry weather was reported by Gregory [2] , Kamal and Singh [4,5] and Kamal and Verma [6]. The lowest aeromycoflora on 25.12.88 may be due to the washing down of atmospheric fungal spores along with dust particles as a result of 12 mm and 5.2 mm of rains on 22.12.88 and 24.12.88 respectively. The immediate decrease in spores after rains was observed by Hirst [3], Meredith [10] and Malliah and Rao [8]. Kumar and Gupta [7], while studying the aerospora over a potato field during September found lowest population of microorganisms. In subsequent seasons with gradual fall in temperature and rise in relative humidity the total population of airspora enhanced continuously. Our observations differ from that of Kumar and Gupta's because rainfall was absent during their period of study. While rainfall played an important role in quantifying the aeromycoflora.

Cladosporium is an ubiquitous fungus and its dominance in the fungal aerospora has been reported from different parts of our country [8, 9, 11]. The presence of Alternaria spores in abundance was obtained by Sreeramulu and Ramalingam [13] in the air over paddy fields. Aspergillus species were found to be present in the air mycoflora through out the period of investigation. Most of the Aspergillus colonies were trapped in the month of November and the number decreased in December and January. This is in concurrence with the report by Kamal and Singh [4].

Table 1 : Percentage contributions of different fungi to total aeromycoflora.

Sr. No.	Name of Organism	1987-88	1988-89
1.	*Acremonium implicatum*	1.77	-
2.	*Alternaria alternata*	10.55	5.68
3.	*A. longipes*	0.40	-
4.	*A. tenuissima*	2.10	0.62
5.	*Aspergillus candidus*	0.66	1.38
6.	*A. flavus*	3.15	4.32
7.	*A. fumigatus*	-	0.06
8.	*A. nidulans*	-	0.42
9.	*A. niger*	2.28	0.86
10.	*A. terreus*	1.02	0.36
11.	*A. versicolor*	0.21	0.42
12.	*Aspergillus sp*	-	0.24
13.	*Bipolaris hawaiiensis*	0.10	-
14.	*B. indica*	0.80	0.13
15.	*Cladosporium cladosporioides*	59.33	57.93
16.	*C. herbarum*	2.10	1.48
17.	*Curvularia lunata*	1.61	2.71
18.	*Emericella nidulans*	0.35	-
19.	*Epicoccum nigrum*	3.30	2.45
20.	*E. purpurascens*	0.35	-
21.	*Fusarium equiseti*	0.94	4.13
22.	*F. moniliforme*	-	0.96
23.	*F. pallidoroseum*	0.21	0.46
24.	*Myrothecium verrucaria*	-	0.30
25.	*Nigrospora sphaerica*	-	0.48
26.	*Penicillium citrinum*	2.36	2.39
27.	*P. cyclopium*	3.05	5.00
28.	*P. oxalicum*	0.03	0.90
29.	*P. purpurogenum*	-	0.36
30.	*Periconia saraswatipurensis*	-	0.24
31.	*Rhizopus oryzae*	0.05	1.42
32.	*Trichothecium roseum*	1.94	1.30
33.	White sterile form	1.19	2.06
34.	Black sterile form		0.13

Table 2 : Diurnal variations in the density of total aeromycoflora per 100 cm^3 over potato fields in 1987-88 crop season.

Sampling time	**Sampling numbers and date**			
	I **5.11.87**	**II** **30.11.87**	**III** **26.12.87**	**IV** **21.1.88**
06.00 A.M.	29.4	18.5	12.0	7.4
12.00 A.M.	53.4	37.8	84.3	42.5
06.00 P.M.	119.4	120.6	157.4	51.4
12.00 P.M.	39.5	21.3	15.9	44.3

Two-way analysis of variance table

Source of variation	**DF**	**Sum of Squares**	**Mean Square**	**F**
Among means of treatments A	3	2196.662	732.220	1.132
Among means of treatments B	3	21329.182	7109.727	10.995*
Residual	9	5819.595	646.621	

A = Diurnal cycle; B = Time interval; DF = Degree of freedom. * Significant at 0.05 level.

A = Diurnal cycle; B = Time interval; DF = Degree of freedom. * Significant at 0.05 level.

Table 3 : Diurnal variations in the density of total aeromycoflora per 100 cm³ over potato fields in 1988-89 crop season.

Sampling time	**Sampling numbers and date**			
	I	**II**	**III**	**IV**
	5.11.88	**30.11.88**	**25.12.88**	**21.1.89**
06.00 A.M.	17.1	12.0	7.1	7.3
12.00 A.M.	19.4	32.8	11.3	8.8
06.00 P.M.	59.6	117.8	9.5	22.8
12.00 P.M.	23.3	11.6	6.5	11.1

Two-way analysis of variance table

Source of variation	**DF**	**Sum of Squares**	**Mean Square**	**F**
Among means of treatments A	3	3141.09	1047.03	2.105
Among means of treatments B	3	4532.221	1510.740	3.037
Residual	9	4475.879	497.319	

A = Diurnal cycle; B = Time interval; DF = Degree of freedom. * Significant at 0.05 level.

Table 4 : Percentage abundance of aeromycoflora over potato fields in the year 1987-88

S.n.	Name of organism	Sampling Numbers, Date and Sampling time															
		I				I				I7				I			
		5.11.88				30.11.88				25.12.88				21.1.89			
		6AM	12AM	6PM	12PM	6AM	12AM	6PM	12PM	6AM	12AM	6PM	12PM	6AM	12AM	6PM	12PM
1.	Acremonium implicatum	-	5.17	2.32	6.97	-	-	1.52	4.34	-	-	-	-	-	5.43	3.51	-
2.	Alternaria alternata	5.46	8.62	7.75	11.62	14.28	9.75	4.58	17.39	11.53	17.39	9.41	29.41	8.33	16.30	11.60	12.5
3.	A. longipes	2.34	3.44	-	2.32	-	-	-	-	-	-	-	-	-	-	-	-
4.	A tenuissima	-	-	2.32	6.97	-	2.43	1.52	-	-	2.17	1.76	-	-	4.34	3.57	3.12
5.	Aspergillus candidus	-	-	0.77	-	-	-	-	-	-	-	-	11.76	8.32	2.17	2.67	-
6.	A. flavus	8.59	5.17	3.10	4.65	14.28	7.31	1.52	13.04	-	-	1.76	11.76	-	-	0.89	2.08
7.	A. niger	3.90	3.44	1.55	6.97	4.76	2.43	2.29	8.69	7.69	1.08	1.17	5.88	-	-	0.89	1.04
8.	A. terreus	1.56	1.72	-	-	4.76	4.87	0.76	-	3.84	2.17	-	-	-	1.08	-	2.08
9.	A. versicolor	-	-	0.77	-	-	-	0.76	-	-	-	-	-	-	-	-	-
10.	Bipolaris hawaiiensis	-	1.72	-	-	-	-	-	-	-	-	-	-	-	-	-	-
11.	B. indica	-	1.72	-	-	-	2.43	3.05	-	-	1.08	-	-	-	-	0.89	-
12.	Cladosporium cladosporioides	46.87	53.44	69.76	37.20	42.85	56.09	67.93	39.13	53.84	59.78	70.58	35.29	20.83	56.52	48.21	54.16
13.	C. herbarum	-	-	-	-	-	-	5.34	-	-	-	4.70	-	-	-	4.46	4.16
14.	Curvularia lunata	-	-	-	-	-	-	1.52	-	-	4.34	2.35	-	-	2.17	-	5.20
15.	Emericella nidulans	-	-	-	-	-	-	-	-	3.84	-	-	-	4.16	-	3.57	2.08
16.	Epicoccum nigrum	-	-	2.32	2.32	-	-	2.29	4.34	7.69	1.08	2.35	-	33.33	9.78	8.92	9.37
17.	E. purpurascens	-	-	-	-	-	-	-	-	-	3.26	-	-	4.16	-	-	-
18.	Fusarium equiseti	8.59	-	2.32	4.65	4.76	-	-	-	-	-	-	-	-	-	-	-
19.	F. pallidoroseum	3.12	-	0.77	-	-	-	-	-	-	-	-	-	-	-	-	-
20.	Penicillium citrinum	8.59	6.89	3.87	9.30	4.76	2.43	-	8.69	3.64	-	-	-	8.32	-	-	2.08
21.	P. cyclopium	3.12	3.44	1.55	-	9.52	7.31	3.05	-	7.69	4.34	4.11	-	-	1.08	2.67	-
22.	P. oxalicum	-	-	-	-	-	-	-	-	-	-	-	-	4.16	-	-	-
23.	Rhizopus oryzae	1.56	-	-	-	-	-	-	-	-	-	-	-	-	- -	-	-
24.	Trichothecium roseum	3.90	3.44	-	2.32	-	4.87	3.05	-	-	3.26	1.17	-	4.16	-	4.46	-
25.	White sterile form	2.34	1.72	0.77	4.65	-	-	0.76	4.34	-	-	0.58	5.88	4.16	1.08	0.89	2.08

Table 5 : Percentage abundance of aeromycoflora over potato fields in the year 1987-88

S.n.	Name of organism	Sampling Numbers, Date and Sampling time															
		I				I				17				I			
		5.11.87				30.11.87				25.12.87				21.1.88			
		6AM	12AM	6PM	12PM	6AM	12AM	6PM	12PM	6AM	12AM	6PM	12PM	6AM	12AM	6PM	12PM
1.	Alternaria alternata	1.35	3.57	6.20	2.24	-	9.85	7.36	4.00	-	4.08	2.43	3.57	3.12	3.47	4.04	12.50
2.	A tenuissima	-	-	1.28	-	-	0.70	0.73	-	-	-	-	-	-	-	1.01	2.08
3.	Aspergillus candidus	1.35	-	-	-	-	2.11	-	-	6.45	-	4.87	7.14	12.5	10.52	4.04	2.08
4.	A. flavus	-	1.19	1.02	3.37	30.76	7.74	1.28	10.00	25.80	2.04	-	21.42	18.75	-	-	10.41
5.	A. fumigatus	-	-	-	-	-	0.70	-	-	-	-	-	-	-	-	-	-
6.	A. nidulans	-	-	-	-	-	1.40	0.36	6.00	-	-	-	-	-	-	-	-
7.	A. niger	-	-	0.51	8.98	1.92	-	0.55	-	3.22	-	-	-	-	-	-	-
8.	A. terreus	-	-	-	-	-	-	0.92	-	-	-	-	-	3.12	-	-	-
9.	A. versicolor	5.40	-	-	-	3.37	-	-	-	-	-	-	-	-	-	-	-
10.	Aspergillus sp.	-	-	-	-	1.92	-	-	-	-	-	-	-	3.12	-	-	-
11.	Bipolaris indica	-	-	-	-	-	-	-	-	-	-	-	-	-	3.47	1.01	4.16
12.	Cladosporium cladosporioides	39.18	64.28	65.11	56.17	40.38	59.15	72.00	32.00	19.35	44.89	63.41	53.57	31.25	28.00	37.37	43.75
13.	C. herbarum	5.40	-	-	-	-	-	0.92	-	-	-	2.43	-	-	17.47	6.06	4.16
14.	Curvularia lunata	5.40	3.57	1.02	-	-	6.33	3.68	6.00	-	4.08	-	-	-	3.47	-	-
15.	Epicoccum nigrum	-	-	0.51	-	-	1.40	0.73	-	16.12	20.40	-	-	-	3.47	13.13	8.33
16.	Fusarium equiseti	8.10	7.14	8.77	6.74	1.92	4.22	2.76	-	-	-	-	-	-	-	6.06	-
17.	F. moniliforme	-	1.19	1.53	-	-	2.11	1.47	-	-	-	-	-	-	-	-	-
18.	F. pallidoroseum	2.70	1.19	-	-	-	-	0.36	-	-	-	-	-	-	6.94	-	-
19.	Myrothecium verrucaria	-	-	-	-	-	-	-	-	-	-	2.43	-	-	-	4.04	-
20.	Nigrospora sphaerica	-	2.38	1.55	-	-	1.40	-	-	-	-	-	-	-	-	-	-
21.	Penicillium citrinum	13.5	8.33	4.12	8.98	-	-	0.18	4.0	-	-	-	-	3.12	-	-	-
22.	P. cyclopium	-	3.57	-	4.49	7.69	2.11	2.20	12.00	16.12	14.28	9.75	7.14	12.5	21.00	20.20	2.08
23.	P. oxalicum	10.81	-	-	2.24	5.76	-	-	-	-	-	4.87	-	-	-	-	-
24.	P. purpurogenum	-	-	-	-	-	-	0.18	6.00	-	-	-	-	6.25	-	-	-
25.	P. saraswatipurensis	-	-	-	-	-	-	-	-	-	8.16	-	-	-	-	-	-
26.	Rhizopus oryzae	-	-	1.02	-	7.69	8.70	1.47	2.00	6.45	2.04	7.13	-	3.12	-	-	-
27.	Trichothecium roseum	-	1.19	1.02	4.49	-	-	1.28	2.00	-	-	-	-	3.12	-	3.03	4.16
28.	White sterile form	6.75	2.38	3.61	1.12	-	-	0.73	16.00	6.45	-	-	7.14	-	-	-	2.08
29.	Black sterile form	-	-	0.51	-	-	-	-	-	-	-	4.87	-	-	-	-	-

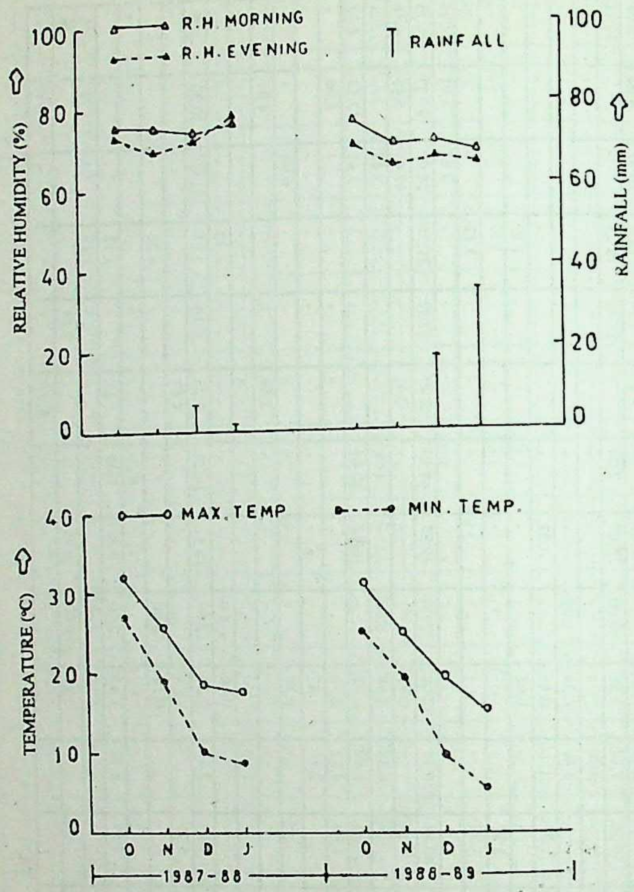

Fig.1 : Meteorological data of Kurukshetra during October, 1987 to January, 1988 and October, 1988 to January, 1989

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Journal of Natural & Physical Science Vol. 9-10 (1995-96) pp 73-76

A NOTE ON RAINGUSH PHENOMENON

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(Received 15-09-96 and after rivision 15-01-97)

Raingush is one of the important meteorological phenomenon. It is not very commonly observed in plains but relatively more frequent in hilly regions. During raingush we have abnormally high rain intensity many times more than that normally observed during heavy rain fall. It is more oftenly observed just after the lightning [1]. Though the scientific observations are not available on meteorological parameters proceeding and preceding raingush, Levin and Ziv [2] have related electric field variation at the time of lightning with occurrence of raingush. Thus it may be worthwhile to look for electrical processes of thunderclouds before, during, and after the lightning discharge.

Many electrical phenomenon those occur in the atmosphere can be explained on the basis of charging mechanism, e.g. airplanes speeding through the atmosphere are subjected to ventilation by ions and will charge to equilibrium potential of several thousand volts [6]. The earth surface is constantly ventilated by winds containing natural ions and is charged accordingly. This is perhaps thought to be one of the possible sources of atmospheric electricity [5]. Cloud drops become charged when they fall through the ionized atmosphere or are ventilated by strong updrafts winds that accompany the growth of convective clouds. Levin and Ziv [2] used a time dependent model coupling the growth of the particles in a cloud with the electrical development. To explain the phenomenon of raingush, they have assumed that the polarization charging is the only mechanism for cloud electrification. According to this mechanism , all the hydrometeors are polarized by downward fair weather electric field. They have negative charge in lower portions and the positive charge in the upper ones because of the gravitational separation. (Fig. 1) . The bigger drops reach the lower parts of the cloud with a net negative charge while the smaller droplets with net positive charge remain in upper region of

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the cloud. Thus a net positive charge is concentrated at the top of the cloud and a net negative charge become concentrated at the bottom. These concentrations cause an increase in the electric field which results in larger charge separation. This process continues until the electric field becomes strong enough to oppose the gravitational forces which help in the floatation of the particles or decrease their fall velocities.

In an electrified cloud a sudden growth of precipitation is often observed in association with lightning. Apparently, there seems to be a close association between the occurrence of lightning, a rapid growth of precipitation particles in the cloud (as may be concluded from the intensification of radar echo) and the sudden onset of a heavy precipitation at the ground beneath the cloud few minutes after the lightning [3,4]. This provides a basis for the hypothesis that the coalescence of cloud droplets into precipitating particles in the form of rain or hail, may be largely enhanced by the electric field and high charge densities in the vicinity of lightning channel. In the above context, Levin and Ziv [2] shown that the precipitating particles of the size of 200 μ m when within the cloud particles of the constant size of 10 μ m grow to 300 μ m in a time of about 300 sec. Thus the fall velocities also increase. Simultaneously, there is an increase in the charge on the particles and electrical field as shown by them {see fig 2(a), (b) reproduced herewith the permission}. Both the growth of charge on the particles and the electric field decrease the terminal velocity of large particles which are negatively charged. This also reduces the collision rate and rate of collection of smaller particles. This decreased number of collisions results in a reduced rate of charge accumulation on the large particles as well as a reduction in the rate of growth of the electric field. During the growth of the electric field, the precipitation rate increases as the result of an increase in both the radii of their drops and their fall velocities [Fig. 2(a)].

When the lightning occurs, a large fraction of charge is neutralized and electric field is decreased considerably. Thus the electrical forces those are responsible to hold the precipitation particles together are weakend drastically i.e. the electric field is not large enough to suspend them against the gravitational force, hence the droplets fall down suddenly in the form of rain, resulting a dramatic increase in the precipitation rate. Thus the raingush is observed. The increase in the precipitation rate is followed by further interaction among the particles, more charge separation, and a subsequent fast recovery of the field. The increase in the field strength again slows down the fall velocities of the particles and reduces precipitation rate. This reduction continues until the field again reaches the lightning threshold and results in second raingush.

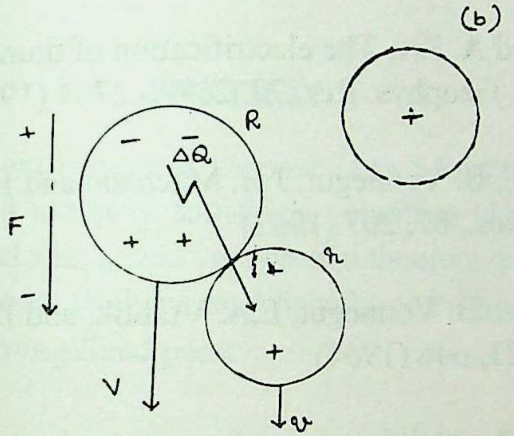

Fig. 1 Sequence of events during the polarization charging of cloud hydrometeors. (a) During contact (b) after separation.

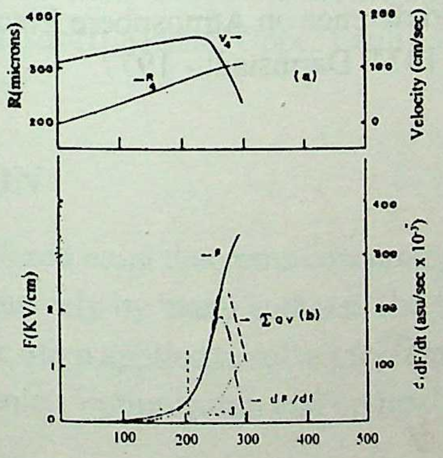

Fig. 2 Changes with time (a) The radius of precipitation particles R and their fall velocities v (b) The electric field F and rate of change of electric field $\frac{dF}{dt}$

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Journal of Natural & Physical Science Vol. 9-10 (1995-96) pp 77-86

FIXED POINT THEOREMS FOR MULTIFUNCTIONS

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(Received 10-10-96)

ABSTRACT

In the present paper our purpose is two-fold relating to multivalued mapping satisfying condition [1]; first we have proved a localized version of a theorem due to ciric [2] and second, we have established a convergence theorem concerning fixed point.

Mathematics Subject Classification (1991) :
Primary - 54H25; Secondary - 47H10
Key words : Orbitally Complete Metric Space, Set Valued Mapping, Hausdorff Metric, Upper Semi Continuous Function, Fixed Point.

INTRODUCTION

The study of fixed point theorems concerning multifunctions has been developed extensively by many authors. The fixed point theorems for multifunctions are often applied to solve problems in connection with mathematical economics, optimization and optimal control [4, 8].

The developments of geometric fixed point theory for multifunctions were initiated by Nadler, Jr. [7] and subsequently pursued by several authors [1, 3, 4, 5, 6, 8, 11]. The analysis for multifunc-

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tion is primarily based on the concept of convergence of sets.

PRELIMINARIES

Let (M, d) be a metric space and let $C(M) = \{A \in 2^M : A$ is a closed subset of $M\}$.

Define the distance between ay two members E, F in $C(M)$ by

$$H(E, F) = \max\left\{\sup_{y \in E} D(y, F), \sup_{z \in F} D(z, E)\right\}$$

Where $D(x, A) = \inf_{y \in A} d(x, y)$, for any non-empty subset A of M. This H is called the Hausdorff metric in $C(M)$; it can be characterised by

$$H(E, F) = \sup_{x \in M} |D(x, E) - D(x, F)| \text{ and also}$$

$$H(E, F) = \inf\{r \mid E \subseteq F_r \text{ and } F \subseteq E_r\}$$

where $A_r = \{x \mid D(x, A) < r\}$ for $r > 0$.

We write $E_n \xrightarrow{H} E$ (as $n \to \infty$) or $E = H\text{-}\lim E_n$ for the sequence $\{E_n\}_n$ in $C(M)$ converging to E in $C(M)$, with respect to the Hausdorff metric H

i.e. $H(E_n, E) \to 0$ (as $n \to \infty$) or $E = \left(\bigcap_{n \geq 1}\right) cl\left(\bigcap_{k < n} E_k\right)$

If M is complete, then it can be shown that $(C(M), H)$ is a complete metric space.

Let X and Y be two sets. A multifunction $F : X \to 2^Y$ is a function from X into the power set 2^Y of Y, that is, for each x in X, Fx is a subset of Y.

For multifunction, F, the notion of fixed point has been modified. A point x in X is said to be a fixed point of F if and only if x is contained in its F-image i.e. $x \in Fx$.

A multifunction $F : M \to C(M)$ is called a multivalued contraction mapping if and only if there exists a fixed q. $0<q<1$, such that

$$H(Fx\,,Fy) \leq q\ d(x\,,y), \text{ for all } x\,,y \text{ in } M.$$

Recently, Ciric [2] proved fixed point theorems for multifunction F on M which are not necessarily continuous and which satisfy a condition of type,

(1) $\min\{\,H(Fx\,,Fy),\,D(x\,,Fx),\,D(y\,,Fy)\} - \min\{\,D(x\,,Fy),\,D(y\,,Fx)\} \leq q\,d(x\,,y)$ for some $0 < q < 1$ and for all $x\,,y$ in M.

In doing so he generalised some results of Nadler [7], Dube and Singh [3] concerning multivalued contraction mappings. In order to establish these extension he introduced the notion of orbit of multifunction F and F-orbitally complete metric space.

An orbit of F at the point x in M is a sequence $\{x_n : x_n \in F\,x_{n-1}\}$.

A space M is said to be F - orbitally complete if and only if every Cauchy sequence of the form $\left\{x_{n_i} : x_{n_i} \in Fx_{n_i - 1}\right\}$ converges in M.

A multifunction F is said to be upper semi-continuous if for any sequences $\{x_n\}_n$ and $\{y_n\}_n$ in M, $y_n \in F\,x_n$, the conditions $\lim x_n = x$ and $\lim y_n = y$ would imply $y \in Fx$.

F is said to be lower semi-continuous if for any sequence $\{x_n\}_n$ in M there exists a sequence $\{y_n\}_n$ with $y_n \in Fx_n$ such that the condition $\lim x_n = x$ and $\lim y_n = y$ would imply $y \in Fx$.

Finally, F is continuous if it is both lower and upper semi continuous.

A multifunction F is orbitally upper semi continuous, for any orbit $\{x_n\}$ of F at some x in M

$$x_n \to u^{*} \in\ M \Rightarrow u \in F\,u.$$

The objective of this paper is two-fold. First, we prove a localized version of Theorem - 5 of Ciric [2] and secondly, we investigate a convergence theorem concerning fixed points satisfying condition [1] to multivalued mappings.

THEOREM 1. Let $B = B(x_0, r) = \{ x \in M : d(x_0, x) < r\}$ for any $x_0 \in M$ and $r > 0$ where (M, d) is an orbitally complete metric space. Let $f : B \to C(M)$ be an orbitally upper semi-continuous multifunction satisfying condition 1] for x, y in B and 2] $D(x_0, Fx_0) < (1 - q)r$. Then F has a fixed point.

PROOF. It follows from condition - 2] of the theorem that there exists x_1 in Fx_0 such that $d(x_0, x_1) < (1 - q)r$ and so $x_1 \in B$.

We may assume that $H(Fx_0, Fx_1) > 0$, otherwise if follows that $Fx_0 = Fx_1$ yielding that $x_1 \in Fx_1$ i.e. x_1 is a fixed point of F. Now from condition 1],

$$\min\{H(Fx_0, Fx_1), D(x_0, Fx_0), D(x_1, Fx_1)\} - \min\{D(x_0, Fx_1), D(x_1, Fx_0)\}$$
$$\leq qd(x_0, x_1).$$

which implies that $\min\{H(Fx_0, Fx_1), D(x_0, Fx_0), D(x_1, Fx_1)\} \leq qd(x_0, x_1)$.

Since $x_1 \in Fx_0$, therefore, $D(x_1, Fx_1) \leq H(Fx_0, Fx_1)$ whence it follows that

3] $\min\{D(x_0, Fx_0), D(x_1, Fx_1)\} \leq qd(x_0, x_1)$.

If $\min\{D(x_0, Fx_0), D(x_1, Fx_1)\} = D(x_0, Fx_0)$ then from (3) it follows that

$$\{D(x_0, Fx_0) \leq qd(x_0, x_1).$$

Now choose a real number a with $0 < a < 1$ such that

$$q^{-a} D(x_0, Fx_0) = (1-q)r.$$

Then, for $x_1 \in Fx_0$ we have $d(x_0, x_1) < q^{-a} D(x_0, Fx_0) \leq q^{1-a} d(x_0, x_1)$

which is absurd as $q^{1-a} < 1$.

Thus, from 3] it follows that $D(x_1, Fx_1) \leq qd(x_0, x_1) < q(1-q)r$ which implies that there exists $x_2 \in Fx_1$ such that

$d(x_1, x_2) < q(1-q)r$.

Now, $d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) < (1-q)r + q\,(1-q)r = (1+q)\,(1-q)r$

$$= (1-q^2)\,r < r$$

Hence $x_2 \in B$.

Let us assume that $d(x_0, x_n) < (1+q+q^2) + \ldots + q^{n-1})(1-q)r = (1-q^n)r < r$

and that $d(x_{n-1}, x_n), < q^{n-1}(1-q)r$ with $x_n \in Fx_{n-1}$.

Now, we have that

4] $\min\{H(Fx_{n-1}, Fx_n), D(x_{n-1}, Fx_{n-1}), D(x_n, Fx_n)\} - \min\{D(x_n, Fx_{n-1}),$
$D(x_{n-1}, Fx_n)\} \leq qd\,(x_{n-1}, x_n)$.

Since $x_n \in Fx_{n-1}$, we have $D(x_n, Fx_n) \leq H(Fx_{n-1}, Fx_n)$ and
$D(x_n, Fx_{n-1}) = 0$. Therefore 4] reduces to

5] $\min\{D(x_{n-1}, Fx_{n-1}), D(x_n, Fx_n)\} \leq qd(x_{n-1}, x_n)$.

If $\min\{D(x_{n-1}, Fx_{n-1}), D(x_n, Fx_n)\} = D(x_{n-1}, Fx_{n-1})$ then from 5],

$D(x_{n-1}, Fx_{n-1}) \leq qD(x_{n-1}, x_n)$

Let a be a real number with $0 < a < 1$ such that $d(x_{n-1}, x_n) < q^{-a}D\,(x_{n-1}, Fx_{n-1})$ whence it follows that $d(x_{n-1}, x_n) < q^{1-a}\,d(x_{n-1}, x_n)$ which is absurd since $q^{1-a} < 1$.

Thus, $D(x_n, Fx_n) \leq qd\,(x_{n-1}, x_n) < q^n\,(1-q)r$, which implies that there exists $x_{n+1} \in Fx_n$ such that
$d(x_n, x_{n+1}) < q^n\,(1-q)r$.

$$\begin{aligned} \text{Again, } d(x_0, x_{n+1}) &\le d(x_0, x_n) + d(x_n, x_{n+1}) \\ &< (1+q+q^2+\ldots+q^{n-1})(1-q)r + q^n(1-q)r \\ &= (1+q+q^2+\ldots+q^n)(1-q)r \\ &= (1-q^{n+1})r < r \end{aligned}$$

which yields that $x_{n+1} \in B$. Thus, every member of the sequence $\{x_n\}_n$ is contained in B.

Now, for any $p \in N$ [= the set of naturals],

$$d(x_n, x_{n+p}) \le \sum_{i=1}^{p} d(x_{n+i-1}, x_{n+i}) < \sum_{i=1}^{p} q^{n+i-1}(1-q)r = q^n(1-q^p)r < q^n r \to 0$$

$$as\ n \to \infty.$$

This shows that the sequence $\{x_n\}_n$ is a Cauchy sequence. Since $x_{n+1} \in Fx_n$ (for $n \in N \cup \{0\}$) and M is orbitally complete, there is a point $u \in M$ such that $\lim_{n} x_n$ By the orbitally upper semicontinuity of F, we have $u \in Fu$.

Thus, u is a fixed point of F.

This completes the proof of the theorem.

Now, we shall consider a family of multifunctions satisfying condition 1] and in the next theorem, we shall prove the existence of common fixed point of a pair of such multifunctions.

Theorem 2: Let (M, d) be *a* complete metric space and let $F_i : M \to CB(M)$ $(i=1)$ be two u.c.s. multivalued mappings satisfying condition

1*] $\min\{H(F_1x, F_2y), D(x, F_1x), D(y, F_2y)\}$

$- \min\{D(x, F_2y), D(y, F_1x),\} \le qd(x,y),$

for all $x, y \in M$ where $CB(M)$ denotes the family of all closed bounded subsets of M. Then F_1 and F_2 have a common fixed point in M.

Proof Select $x_0 \in X$ and $x_1 \in F_1x_0$. Assume that $H(F_1x_0 , F_2x_1) \neq 0$, otherwise F_1 and F_2 have a common fixed point. Let $a \in (0, 1)$ be arbitrary.

Then there exists $x_2 \in F_2x_1$ such that $d(x_1,x_2) < q^{-a} D(x_1,F_2x_1)$ and there exists $x_3 \in F_1x_2$ such that $d(x_2,x_3) < q^{-a} D(x_2,F_1x_2)$. Continuing in this way we can obtain a sequence $\{x_n\}_n$ in X with

$x_{2n+1} \in F_1x_{2n}$ and $x_{2n+2} \in F_2x_{2n+1}$ such that

5] $d(x_{2n}, x_{2n+1}) < q^{-a} D(x_{2n},F_1x_{2n})$ and

$d(x_{2n+1}, x_{2n+2}) < q^{-a} D(x_{2n+1},F_2x_{2n+1})$, for all $n \in N \cup \{0\}$.

Now, we have

$$\min\{H(F_1x_{2n},F_2x_{2n+1}), D(x_{2n},F_1x_{2n}),D(x_{2n+1},F_2x_{2n+1})\} -$$

$$\min\{D(x_{2n},F_2x_{2n+1}), D(x_{2n+1},F_1x_{2n})\} \leq qd(x_{2n}, x_{2n+1}).$$

Since $x_{2n+1} \in F_1x_{2n}$, it clearly follows that

$D(x_{2n+1}, F_2x_{2n+1}), \leq H(F_1 x_{2n} ,F_2x_{2n+1})$ and $D(x_{2n+1}, F_1x_{2n}) = 0$ and hence we have

$$\min\{D(x_{2n},F_1x_{2n}), D(x_{2n+1},F_2x_{2n+1})\} \leq qd(x_{2n}, x_{2n+1}). \text{ i.e}$$

6] $\min\{q^{-a} D(x_{2n},F_1x_{2n}), q^{-a} D(x_{2n+1},F_2x_{2n+1})\} \leq q^{1-a} d(x_{2n}, x_{2n+1})$.

If $\min\{q^{-a} D(x_{2n},F_1x_{2n}), q^{-a} D(x_{2n+1},F_2x_{2n+1})\} = q^{-a} D(x_{2n}, F_1x_{2n})$.

then from 5] and 6] it follows that

$d(x_{2n},x_{2n+1}) < q^{1-a} d(x_{2n},x_{2n+1})$ which is absurd since $q^{1-a} < 1$.

Therefore, $q^{-a} D(x_{2n+1},F_2x_{2n+1}) < q^{1-a} d(x_{2n},x_{2n+1})$.

form 5] it follows that $d(x_{2n+1},x_{2n+2}) < q^{1-a} d(x_{2n}, x_{2n+1})$.

Repeating this argument $2n$-times, we obtain,

$$d(x_{2n+1}, x_{2n+2}) < q^{1-a}\, d(x_{2n}, x_{2n+1}) < \ldots < (q^{1-a})^{2n+1}\, d(x_0, x_1).$$

Similarly, $d(x_{2n}, x_{2n+1}) < (q^{1-a})^{2n}\, d(x_0, x_1)$.

Since $q^{1-a} < 1$, by routine calculation it can be easily seen that the sequence $\{x_n\}$ is a Cauchy sequence and hence it will converge to some $u \in M$ i.e. $\lim\limits_{n} x_n = u \cdot$

Since $x_{2n+1} \in f_1 x_{2n}$ and $\lim\limits_{n} x_{2n+1} = u = \lim\limits_{n} x_{2n}$, then the upper semicontinuity of F_1 yields $u \in F_1 u$ i.e. u is a fixed point of F_1. Similarly, it can be shown that $u \in F_2 u$.

Hence u is a common fixed point of F_1 and F_2.

Theorem 3: Let (M, d) be a complete metric space and E a compact subset of M. Let for each $n \in N$ $T_n : E \to CB(M)$ be continuous multifunction satisfying condition

$$\min\{H(T_n x, T_{n+1} Y), D(x, T_n x), D(y, T_{n+1} y)\} - \min\{D(x, T_{n+1} y), D(y, T_n x),\}$$

$$\leq q d(x,y),$$

for all $x, y \in E$ and for some $q \in (0, 1)$. Let $T : E \to CB(M)$.

If $\{T_n\}_n$ converges pointwise to T in the sense of the Hausdorff metric, then T has a fixed point. Indeed, Every cluster opint of the sequence $\{u_n\}_n$ of fixed point u_n of T_n is a fixed point of T.

Proof: By theorem 2, for each $n \in N$, T_n and T_{n+1} have a common fixed point, say u_n in E, i.e. $u_n \in T_n u_n$ and $u_n \in T_{n+1} u_n$.

Thus, we have a sequence $\{u_n\}_n$ of fixed points in E and E being compact, there exists a subsequence $\{u_{n_i}\}_i$ of $\{u_n\}_n$ which converges to some $u \in E$.

Let $\varepsilon > 0$ and choose an integer $p > 0$ such that for all $i \geq p$

$$H(T_{n_i}, u, Tu) < \frac{\varepsilon}{3} \text{ and } d(u_{n_i}, u) < \frac{\varepsilon}{3}$$

Since for each $n \in N$, T_n being continuous, for all $i \geq p$,

$$H(T_{n_i} u_{n_i}, T_{n_i} u) < \frac{\varepsilon}{3}$$

Thus, we have

$$\begin{aligned} D(u, Tu) &\leq d(u, u_{n_i}) + D(u_{n_i}, Tu) \\ &\leq d(u, u_{n_i}) + H(T_{n_i} U_{n_i}, Tu) \\ &\leq d(u, u_{n_i}) + H(T_{n_i} U_{n_i}, T_{n_i} u) + H(T_{n_i} u, Tu) < \varepsilon \end{aligned}$$

Hence $D(u, Tu) = 0$ and hence $u \in \overline{Tu} = Tu$ as $Tu \in CB(M)$, i.e. u is a fixed point of T.

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Journal of Natural & Physical Science Vol. 9-10 (1995-96) pp 87-94

PROBLEM OF LINE EXPLOSION IN A GAS CLOUD

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(Received 12-08-96)

ABSTRACT

In the present paper similarity solutions have been developed describing the propagation of cylindrical shock in non-uniform atmosphere taking counter gas pressure and radiation heat flux into account. The total energy of the explosion is constant.

INTRODUCTION

Lin (1954) obtained numerically the solution for cylindrically symmetric flow. Ray (1957) duscussed the problems of point and line explosion and found an exact analytic solution. Analytic solutions in the three cases of plane, cylindrical and spherical flow have been noted by Sakurai (1955). Rogers (1958) has also studied the similarity solutions for three cases in uniform atmosphere. Similarity solutions describing the flow of a perfect gas behind cylindrical shock waves with radiation heat flux are investigated.

The problem of propagation of shock waves in a non-homogeneous medium is of great interest in exploring the effect of explosion in the stars and atmosphere of the earth.

The radiation pressure and ratiation energy have been ignored.

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The gas in the undisturbed field is assumed to be at rest. We have also assumed the gas to be gray and opaque and the shock to be transparent and isothermal.

The total energy of the expanding wave has been considered to remain constant. The solution, however, is only applicable to a gaseous medium where the undisturbed pressure falls as the inverse square of the distance from the line of explosion.

BASIC EQUATIONS AND BOUNDARY CONDITIONS

The equation of conservation of mass, momentum and energy behind the wave are

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial r} + \rho \frac{\partial u}{\partial r} + \frac{\rho u}{r} = 0 \qquad (2.1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0 \qquad (2.2)$$

$$\frac{\partial e}{\partial t} + u \frac{\partial e}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial t} + pu \frac{\partial}{\partial r} \left(\frac{1}{\rho} \right) + \frac{1}{\rho r} \frac{\partial}{\partial r} (rq) = 0 \qquad (2.3)$$

where ρ is the density, p, the pressure, u, the radial velocity, q, the heat flux, t, the time and E, the internal energy.

For an ideal gas,

$$e = \frac{p}{(\gamma - 1)\rho}, \qquad p = \Gamma \rho T \qquad (2.4)$$

where γ is the adiabatic gas index, T, the temperature, and Γ the gas constant.

Considering local thermodynamic equilibrium and taking Rosseland's diffusion approximation. We have

$$q = \frac{-c\mu}{3} \frac{\partial}{\partial r} (\sigma T^4) \qquad (2.5)$$

Where $\frac{1}{4}\sigma c$ is the Stefan-Boltzmann constant, c, the velocity of light, and μ, the mean-free path of radiation, is a function of density and temperature.

Following Wand (1966), We take

$$\mu = \mu_0 \rho^\alpha T^\beta \tag{2.6}$$

μ, ρ and β are constants. The distrubance is headed by an isothermal shock and the conditions are

$$\rho_2(v - u_2) = \rho_1 v = m_n \tag{2.7}$$

$$p_2 - p_1 = m_s u_2 \tag{2.8}$$

$$e_2 \frac{p_2}{\rho_2} + \frac{1}{2}(v - u_2)^2 - \frac{q_2}{m_s} = e_1 + \frac{p_1}{\rho_1} + \frac{1}{2}v^2 \tag{2.9}$$

$$T_2 = T_1 \tag{2.10}$$

Where subscripts 1 and 2 are for the regions just ahead and just behind of the surface, respectively and v denotes the shock velocity.

In front of the shock in the undisturbed gaseous medium, the dendity and pressure distributions are as given below.

$$p_1 = AR^n \qquad n < 0 \tag{2.11}$$

$$\rho_1 + BR^\omega \qquad -2 < \omega < 0 \tag{2.12}$$

Where R is the shock radions $A.B.$ n and ω areconstants.

Now let us consider solutions of the equations (2.7) - (2.10) in the form

$$u = \frac{r}{t} v(\eta)$$

$$\rho = r^K t^\lambda \Omega(\eta)$$

$$p = r^{K+2}\, t^{\lambda-2}\, p(\eta) \tag{2.13}$$

$$q = r^{K+3}\, t^{\lambda-3}\, F(\eta)$$

where $\eta = r^{a} + t^{b}$ and λ, K, a and b are constants (2.14)

The total energy of the disturbance per unit length is

$$E = 2\pi \int_{0}^{R} \left(\frac{1}{2}\rho\, u^{2} + \frac{p}{\gamma-1}\right) r\,dr \tag{2.15}$$

In terms of variable η we get

$$E = \frac{2\pi}{a} \int_{\infty}^{\eta_0} \left(\frac{1}{2}\Omega v^{2} + \frac{p}{\gamma-1}\right) \eta^{[(k+4)/a-1]t[\lambda-2-(a/b)(k+4)]}\, d\eta \tag{2.16}$$

where η_0 is the value of η at the shock front.

We take the shock surface to be given by η_0 = constant and velocity of the shock surface as

$$v = \frac{-b}{a}\,\frac{R}{t} \tag{2.17}$$

The total energy of disturbance within the shock surface at any time t is constant. This by (2.16) requires that

$$\lambda - 2 - \frac{b}{a}(K+4) = 0 \tag{2.18}$$

Suppose the Mach mamber at the shock front be defined by

$$M^{2} = \frac{\rho_1 v^{2}}{r p_1}$$

Substitution of (2.13) in the equation of motion (2.1) to (2.6), shock conditions (2.7) to (2.10) and Equation (2.19) and after using the relation (2.18) we obtain that similarity conditions, following Singh and

Vishwakarma (1983), are compatible when

$k = \omega,$ $$\lambda = 0, \qquad a = -(4 + \omega), \qquad b = 2$$

$n = -2,$ $$\beta = \frac{-5}{2}, \qquad and\ \alpha = \frac{\omega + 1}{\omega} \tag{2.20}$$

Therefore, the pressure distribution becomes

$$p_1 = AR^{-2} \tag{2.21}$$

and Equations (2.1) to (2.3) and (2.5) are now sought into the forms

$$\frac{\Omega'(\eta)}{\Omega(\eta)} = \frac{\eta(4+\omega)v'(\eta) - (2+\omega)v(\eta)}{\eta[2-(4+\omega)v(\eta)]} \tag{2.22}$$

$$\frac{P'(\eta)}{P(\eta)} = \frac{\Omega(\eta)}{p(\eta)} \left[\frac{\eta v'(\eta)\{2-(4+\omega)v(\eta)\} + v(\eta)(v(\eta)-1)}{\eta(4+\omega)} \right] + \frac{2+\omega}{\eta(4+\omega)} \tag{2.23}$$

$$\frac{F'(\eta)}{F(\eta)} = \frac{P'(\eta)}{F(\eta)} \left[\frac{2-(4+\omega)v(\eta)}{(\gamma-1)(4+\omega)} \right] + \frac{P(\eta)}{F(\eta)} \left[\frac{(\omega+2\gamma+2)v(\eta) - \gamma(4+\omega)\eta v'(\eta) - 2}{\eta(\gamma-1)(4+\omega)} \right] + \frac{1}{\eta} \tag{2.24}$$

$$\frac{F(\eta)}{P(\eta)} = -N \frac{P(\eta)^{1/2}}{\Omega(\eta)\left(\frac{3}{2}-\alpha\right)} \left[2 - (4+\omega)\eta \left\{ \frac{P'(\eta)}{P(\eta)} - \frac{\Omega'(\eta)}{\Omega(\eta)} \right\} \right] \tag{2.25}$$

where

$$N = \frac{4\sigma c \mu_0}{3\Gamma^{2/3} A^{1-\alpha}}$$

= a nondimensional radiation parameter (2.26)

$$\frac{1}{N}\frac{F(\eta)\Omega(\eta)^{\frac{1}{2}-\alpha}}{(p(\eta))^{\frac{1}{2}}}\left[2-(4+\omega)v(\eta)\right]-\frac{2P(\eta)}{\Omega(\eta)}v^{'}(\eta)$$

$$=\frac{\left[(4+\omega)v(\eta)+\omega\right]-\left[2-(4+\omega)v(\eta)\{v(\eta)(v(\eta)-1)\}\right]}{\eta\left[\{2-(4+\omega)v(\eta)^2\}-(4+\omega)^2\frac{P(\eta)}{\Omega(\eta)}\right]} \tag{2.27}$$

The approximation shock conditions are

$$v(\eta_0)=\frac{2}{(4+\omega)}\left[1-\frac{1}{\gamma M^2}\right] \tag{2.28}$$

$$\Omega(\eta_0) = \gamma M^2 \tag{2.29}$$

$$P(\eta_0) = \frac{4}{(4+\omega)^2} \tag{2.30}$$

$$F(\eta_0) = \frac{1}{2}\left(\frac{2}{4+\omega}\right)^3\left[\frac{1}{\gamma^2 M^4}-1\right] \tag{2.31}$$

Which are the initial value for our numerical calculation, where we assume that $\eta^0 = 1$

RESULTS

Differential equations are numerically solved by the Runge-Kutta Method and the solutions are presented in a convenient form as

$$\frac{u}{u_2}=\left(\frac{\eta_0}{\eta}\right)^{\frac{1}{4+\omega}}\frac{v(\eta)}{v(\eta_0)} \tag{3.1}$$

TABLE - I

First set.

(i) $\gamma = \frac{4}{3}$, $M^2 = 20$, $N = 10, \omega = -1.5$, $\lambda = \frac{1}{3}$

η	$\frac{u}{u_2}$	$\frac{\rho}{\rho_2}$	$\frac{p}{p_2}$	$\frac{q}{q_2}$
1.00	1.00	1.00	1.00	1.00
1.20	0.95	0.73	0.59	1.08
1.40	0.92	0.60	0.37	1.15
1.60	0.90	0.55	0.22	1.19
1.80	0.87	0.70	0.12	1.21

TABLE - II

Second set.

(ii) $\gamma = \frac{5}{3}$, $M^2 = 20$, $N = 100$, $\omega = -1.75$, $\lambda = \frac{3}{7}$

η	$\frac{u}{u_2}$	$\frac{\rho}{\rho_2}$	$\frac{p}{p_2}$	$\frac{q}{q_2}$
1.00	1.00	1.00	1.00	1.00
1.20	0.92	0.70	0.69	0.882
1.40	0.87	0.55	0.53	0.833
1.60	0.83	0.46	0.44	0.815
1.80	0.80	0.40	0.38	0.810
2.00	0.77	0.36	0.34	0.822
2.20	0.75	0.34	0.31	0.832

$$\frac{\rho}{\rho_2} = \left(\frac{\eta_0}{\eta}\right)^{\frac{\omega}{4+\omega}} \frac{\Omega(\eta)}{\Omega(\eta_0)}, \tag{3.2}$$

$$\frac{p}{p_2} = \left(\frac{\eta_0}{\eta}\right)^{\frac{2+\omega}{4+\omega}} \frac{P(\eta)}{P(\eta_0)}, \tag{3.3}$$

and
$$\frac{q_1}{q_2} = \left(\frac{\eta_0}{\eta}\right)^{\frac{3+\omega}{4+\omega}} \frac{F(\eta)}{F(\eta_0)}, \tag{3.4}$$

The numerical results for a certain choice of parameters are reproduced in the form of tables. We calculate our results for the following two set of parameters.

(i) $\gamma = \frac{4}{3}, \quad M^2 = 20, \quad N = 10, \quad \omega = -1.5, \quad \lambda = \frac{1}{3}$

(ii) $\gamma = \frac{5}{3}, \quad M^2 = 20, \quad N = 100, \quad \omega = -1.75, \quad \lambda = \frac{3}{7}$

Nature of the field variables may bee seen through the tables I and II. The IInd set of parameters are more effective on the flow variables rather than the Ist. The radiation parameter N affects the variation of density, pressure and radiation heat flux at its value increases.

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Journal of Natural & Physical Science Vol. 9-10 (1995-96) pp 95-100

GENRAL FIXED POINT THEOREMS
FOR
MAPPINGS IN A HAUSDORFF SPACE

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(Received on 6-07-96 after revision 10-03-97)

ABSTRACT

In this note we establish two general fixed point theorems for selfmaps of a Hausdorff space, and then show that the results of Popa (1983) and Chugh and Rani (1992) are special cases of these theorems.

Mathematic Subject Classifications (1991) : 47H10

Keywords and Phrases: Hausdorff spaces, fixed points, pairs of maps

Sehie Park [4], Troy Hicks and Rhoades [2,3] and Rhoades [6,7] have published papers which establish fixed point theorems, in metric spaces and d-complete spaces, using general principles. We have shown that these general principles then contain a number of fixed point theorems of other authors as special cases. It is the purpose of this note to establish two general fixed point theorems for mappings in a Hausdorff space, and then show that the results of Popa [5] and Chugh and Rani [1] are special cases of these theorems.

The orbit of a point x_0 is defined to be the set $U = \{x_0, Tx_0, \ldots\ldots T^n x_0, \ldots\ldots\}$. T will be called x_0-orbitally continuous if T is continuous on U.

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Theorem 1. Let T be a selfmap of a Hausdorff space X, $f: X \times X \to IR^+$, f continucus and such that

(a) $f(x, y) \neq 0$ for each $x \neq y$,

(b) $f(Tx, T^2x) \leq qf(x, Tx)$ for each $x \in X$, $x \neq Tx$, q satisfying $0<q<1$,

(c) there exists a point $x_0 \in X$ such that $\{T^n x_0\}$ has a convergent subsequence, and

(d) T is x_0-orbitally continuous.

Then T has a fixed point

Proof. Define $x_n = T^n x_0$. We may assume that $x_n \neq x_{n+1}$ for each n, since, otherwise, T has a fixed point.

From condition (b), $f(x_n, x_{n+1}) \leq qf(x_{n-1}, x_n) \leq \ldots\ldots \leq q^n f(x_0, x_1)$. Therefore $\{f(x_n, x_{n+1})\}$ is a monotone decreasing positive sequence with limit 0.

From (c) let $\{x_{nk}\}$ denote the convergent subsequence of $\{x_n\}$ and let z denote the limit. Using (d), $z = \lim_k x_{nk}$ and $Tz = T(\lim_k x_{nk}) = \lim_k T(x_{nk}) = \lim_k x_{nk+1}$. Therefore, using the continuity of f, $f(z, Tz) = f(\lim_k x_{nk}, \lim_k x_{nk+1}) = \lim_k f(x_{nk}, x_{nk+1}) = 0$, since $\lim f(x_n, x_{n+1}) = 0$. From (a), $z = Tz$.

Corollary 1. (Theorem 2 of Popa [5]) Let T be a continuous selfmap of a Hausdorff space, $f: X \times X \to IR^+$, f continuous and such that

(a) $f(x, y) \neq 0$ for each $x \neq y$,

(b) $f(Tx, Ty) \leq \dfrac{af(x,Tx)f(y,Ty)}{f(x,y)} + bf(x, y)$ for all $x \neq y$, $a, b \geq 0$, $a + b < 1$, and

(c) $f^2(x, y) \geq f(x, x)f(y, y)$ for each $x \neq y \in X$.

If, for some $x_0 \in X$, $\{T^n x_0\}$ has a convergent subsequence, then T has a unique fixed point.

Proof. In (b) set $y = Tx$ to obtain condition (b) of Theorem 1. Then, from Theorem 1, T has a fixed point. Condition (c) implies uniqueness.

Corollary 2. (Theorem 3 of Chugh and Rani [1]) Let T be a continuous selfmap of a Hausdorff space, $f: X \times X \to IR^+$, f continuous and such that

(a) $f(x, y) \neq 0$ for each $x \neq y$,

(b) $f(Tx, Ty) \leq \dfrac{\alpha f(y,Ty)[1+f(x,Tx)]}{1+f(x,y)} + \beta f(x, y)$ for all $x \neq y$,

$\alpha, \beta > 0$, $\alpha + \beta < 1$, and

(c) $f(x, y) \geq \dfrac{1+f(x,x)}{1+f(y,y)}$ for all $x, y \in X$.

If, for some $x_0 \in X$ the sequence $\{T^n x_0\}$ has a convergent subsequence, then T has a unique fixed point.

Proof. Set $y = Tx$ in (b) to obtain (b) of Theorem 1. Then T has a fixed point. Uniqueness follows from condition (c).

Theorem 2. Let T_1 and T_2 be selfmaps of a Hausdorff space, $f: X \times X \to IR^+$, f continuous and such that

(a) $f(x, y) = f(y, x)$ for each $x, y \in X$,

(b) $f(x, y) \neq 0$ for each $x \neq y$, $x, y \in X$,

(c) $f(T_1x, T_2y) \leq qf(x, y)$ for each $x \neq y$, $x, y \in \{x_i\}$, $\{x_i \mid x_{2n+1} = T_1x_{2n}, x_{2n+2} = T_2x_{2n+1}\} \subset X$, satisfying either $x = T_2y$ or $y = T_1x$,

(d) there exists a point $x_0 \in X$ such that $\{x_n\}$, as defined in (b), has a-convergent subsequence, and

(e) T_1 and T_2 are $\{x_{2n}\}$, $\{x_{2n+1}\}$, - orbitally continuous, respactively.

Then either T_1 or T_2 has a fixed point or T_1 and T_2 have a common fixed point.

Proof. Suppose that $x_n = x_{n+1}$ for some n. If n is even, then we have $x_{2n} = x_{2n+1} = T_1x_{2n}$, and x_{2n} is a fixed point of T_1. Suppose that n is odd. Then we have $x_{2n+1} = x_{2n+2} = T_2x_{2n+1}$, and x_{2n+1} is a fixed point of T_2.

Suppose that $x_n \neq x_{n+1}$ for each n. then, from (c) and (a), $f(x_{2n+1}, x_{2n+2}) = f(T_1x_{2n}, T_2x_{2n+1}) \leq qf(x_{2n}, x_{2n+1})$ and $f(x_{2n}, x_{2n+1}) = f(T_2x_{2n-1}, T_1x_{2n}) = f(T_1x_{2n}, T_2x_{2n-1}) \leq qf(x_{2n+1}, x_{2n}) = qf(x_{2n}, x_{2n+1})$, so that, for each n, $f(x_n, x_{n+1}) \leq qf(x_{n-1}, x_n)$, which implies that $f(x_n, x_{n+1}) \leq q^n f(x_0, x_1)$, and $\lim f(x_n, x_{n+1}) = 0$.

From condition (d) $\{x_n\}$ has a convergent subsequence. This convergent subsequence has either a convergent subsequence consisting of terms with only even subscripts or a convergent subsequence consisting of terms with only odd subscripts. For definiteness, assume that the convergent subsequence consists of only even subscripts. For notational convenience, denote this subsequence by $\{x_{2m}\}$. The other case is proved similarly. Using condition (d), with $z = \lim x_{2m}$, condition (e) implies that $T_1z = T_1(\lim x_{2m}) = \lim T_1(x_{2m}) = \lim x_{2m+1}$. Therefore, since f is continuous,

$$f(z, T_1z) = f(\lim x_{2m}, \lim x_{2m+1}) = \lim f(x_{2m}, \lim x_{2m+1}) = 0,$$

which, from (b), implies that $z = T_1z$. In a similar nammer it can be shown that $z = T_2z$.

Corollary 3. (Theorem 3 of Popa [5]) Let T_1, T_2 be continuous selfmaps of a Hausdorff space $f : X \times X \rightarrow IR^+$, f continuous and such that

(a) $f(x, y) = f(y, x)$ for each $x, y \in X$,

(b) $f(x, y) \neq 0$ for each $x \neq y \in X$,

(c) $f(T_1x, T_2y) \leq \dfrac{af(x,T_1x)f(y,T_2)}{f(x,y)} + bf(x, y)$ for all $x \neq y \in X$, a, $b \geq 0$, $a + b < 1$, and

(d) $f^2(x, y) \leq f(x, x)f(y, y)$ for all $x, y \in X$.

if, for some $x_0 \in X$, the sequence $\{x_n\}$ defined by $T_1x_{2n} = x_{2n+1}$, $T_2x_{2n+1} = x_{2n+2}$, has a convergent subsequence. Then either T_1 or T_2 has a fixed point or T_1 and T_2 have a unique common fixed point.

Proof. Set $y = T_1x$ in (c) to obtain (b) of Theorem 2. Then T_1 and T_2 have a common fixed point. Uniqueness follows from condition (d).

Remark 1. Note that the restriction on the subsequence is not needed.

Corollary 4. (Theorem 4 of Chugh and Rani [1]) Let T_1, T_2 be continuous selfmaps of a Hausdorff space $f: X \times X \to IR^+$, f continuous and such that

(a) $f(x, y) = f(y, x)$ for each $x, y \in X$,

(b) $f(x, y) \neq 0$ for each $x \neq y \in X$,

(c) $f(T_1x, T_2y) \leq \dfrac{\alpha f(y,T_2y)[1+f(x,T_1x)]}{1+f(x,y)} + \beta f(x, y)$ for all $x \neq y$, $\alpha, \beta > 0$, $\alpha + \beta < 1$, and

(d) $f(x, y) \geq \dfrac{1+f(x,x)}{1+f(y,y)}$ for all $x, y \in X$.

If, for some $x_0 \in X$, the sequence $\{x_n\}$ defined by $T_1x_{2n} = x_{2n+1}$, $T_2x_{2n+1} = x_{2n+2}$, has a convergent subsequence of the type $\{x_{(2p+1)n}\}$, p fixed, then either T_1 or T_2 has a fixed point or T_1 and T_2 have a unique fixed point.

Proof. Set $y = T_1x$ in (c). Then one obtains condition (b) of Theorem 2. Therefore T_1 and T_2 have a common fixed point. Uniqueness follows from (d). As in Corollary 3, the special nature of the convergent subsequence is not needed.

Remark 2. Theorem 3 of Popa [5] and Theorem 4 of Chugh and Rani [1] are incorrectly stated. If $x_n = x_{n+1}$ for some n, one cannot conclude that also $x_{n+1} = x_{n+2}$, and therefore x_n is a common fixed point of T_1 and T_2. The reason for this is that, even though we know, from (b), that $f(x, y) = 0$ implies $x = y$, the converse is not assumed to be true.

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फार्म – 4

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका

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हस्ताक्षर
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Volume 11 (1997)

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका

JOURNAL OF NATURAL & PHYSICAL SCIENCES

गुरुकुल कांगड़ी विश्वविद्यालय, हरिद्वार
Gurukul Kangri Vishwavidyalaya, Hardwar

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका
Journal of Nature & Physical Sciences

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Journal of Natural & Physical Sciences Vol. 11 (1997) 1-12

COMPATIBLE MAPPINGS AND COMMON FIXED POINT FOR FOUR MAPPINGS

P.C. Lohani* and V.H. Badshah **

(Received 28.07.1995)

ABSTRACT

In this paper we prove some common fixed point theorems in metric spaces which extend the results of B.Fisher, G. Jungck, M.S. Khan and M. Imdad.

Mathematics subject Classification (1980) : 54H25
Keywords : Common fixed points, commuting mappings and compatible mappings.

INTRODUCTION

In the year 1976, G, Jungck [5] gave a concept of common fixed point theorem for commuting mappings, which generalizes the well-known Banach's fixed point theorem. Later on, this result was generalized and extended in various ways by many authors : for instance, K.M. Das and K.V. Naik [1] K. Iseki and Bijendra Singh [4], B. Fisher [2], S.P. Singh [14], etc. Recently B. Fisher [3], M.S. Khan and M. Imdad [9], S.M. Kang and Y.P. Kim [8] proved some common fixed point theorems of three and four commuting mappings respectively.

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On the other hand, S. Sessa [12] defined a concept of generalization of commutativity, which is called weak commutativity and gave the common fixed point theorem for weakly commuting mappings, which generalizes the result of K.M. Das and K.V. Naik [1]. Further, G. Jungck [6] introduced more generalized commutativity, called compatibility, which is more general than that of weak commutativity. The utility of compatibility in the context of fixed point theory was initially demonstrated in extending a theorem of S.Park and J.S. Bae [11]. In general, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but the converses are not necessarily true as in the examples of Section 1.

The purpose of this paper is to propose a generalization of some common fixed point theorems, which extend the results of B. Fisher [3]. G. Jungck [7], M.S. Khan and M.Imdad [9] by employing a functional inequality and compatible mappings instead of commuting mappings and also give some examples to illustrate our main theorems.

PRELIMINARIES

The following was introduced by S. Sessa [12].

Definition 1.1 : Mappings S and T from a metric space (X,d) into itself, are called weakly commuting mappings on X, if $d(STx, TSx) \leq d(Sx,Tx)$ for all x in X.

Carearly, commuting mappings are weakly commuting, but the converse is not necessarily true, given by the following example :

Example 1.1 : Let $X = [0,1]$ with the Euclidean metric d. Define S and $T : X \rightarrow X$ by

$$Sx = \frac{x}{2-x} \quad \text{and} \quad Tx = \frac{x}{2}$$

for all x in X. Then we have for any x in X,

$$d(STx, TSx) = \frac{x}{4-2x} - \frac{x}{4-x}$$

$$= \frac{x^2}{(4-2x)(4-x)}$$

$$\leq \frac{x^2}{4-2x} = \frac{x}{(2-x)} - \frac{x}{2}$$

$$= d(Sx, Tx)$$

Thus S and T are weakly commuting mappings on X, but they are not commuting on X since

$$STx = \frac{x}{4-2x} > \frac{x}{4-x} = TSx$$

for any non-zero x in X.

The following was given by G. Jungck [6] and [7] respectively.

Definition 1.2 : Mappings S and T from a metric space (X,d) into itself are said to be compatible mappings on X, if $\lim_{m\to\infty} d(STs_m, TSx_m) = 0$, when $\{x_m\}$ is a sequence in X such that $\lim_{m\to\infty} Sx_m = \lim_{m\to\infty} Tx_m = x$ for some point x in X.

If S and T are compatible mappings on X, then $d(STx, TSx)=0$, when $d(Sx,Tx) = 0$ for some x in X.

Obviously, weakly commuting mappings are compatible, but the converse is not necessarily true as in the following example :

Example 1.2. : Let $x= (-\infty,\infty)$ with the Euclidean metric d. Define S and $T : X \to X$ by

$$Sx = x^2 \quad \text{and } Tx = 2x - 1$$

for all x in X. Since

$$d(Sx_m.Tx_m) = |x_m-1|^2 \to 0, \qquad \text{if } x_m \to 1$$

$$\lim_{m\to\infty} d(Tsx_m, STx_m) = \lim_{m\to\infty} 2|x_m-1|^2 \to 0 \text{ as } x_m \to 1$$

Thus, S and T are compatible on X, but are not weakly commuting mappings on X, since

$d(STx, TSx) = 2 > 1 = d(Sx, Tx)$ for $x=0$ in X.

Using the following Lemma 1.1 and Lemma 1.2 for our main theorems, which were proved by G. Jungck [6] and [7], respectively :

Lemma 1.1 : Let S and T be compatible mappings from a metric space (X,d) into itself. Suppose that $\lim_{m\to\infty} Sx_m = \lim_{m\to\infty} Tx_m = x$ for some x in X. Then $\lim_{m\to\infty} TSx_m = Sx$, if S is continuous.

Now let P,Q,S and T be mappings from a metric space (X,d) into itself satisfying the following conditions :

$$S(x) \subset Q(x) \quad \text{and } T(x) \subset P(x) \tag{1.1}$$

$$d(Sx,Ty) \leq \alpha \frac{d(Qy,Ty)\,[1+d(Px,Sx)]}{[1+d(Px, Qy)]} + \beta d\,(Px,Qy) \tag{1.2}$$

for all x,y in X, where $\alpha,\beta \leq 0$, $\alpha+\beta < 1$. Then for an arbitrary point x_0 in X, by (1.1), we choose a point x_1 in X such that $Qx_1=Sx_0$ and for this point x_1, there exists a point x_2 in X such that $Px_2=Tx_1$ and so on. Proceeding in the similar manner, we can define a sequence $\{y_m\}$ in X such that

$$y_{2m+1}= Qx_{2m+1} = Sx_{2m} \text{ and } y_{2m} = Px_{2m} = Tx_{2m-1} \qquad (1.3)$$

Lemma 1.2 : Let $P.Q.S.$ and T be mappings from a metric space (X,d) into itself satisfying the conditions (1.1) and (1.2). Then the sequence $\{y_n\}$ defined by (1.3) is a Cauchy sequence in X.

FIXED POINT THEOREMS

In this section we prove common fixed point theorems for compatible mappings using functional inequality in complete metric spaces, we have the following :

Theorem 2.1 : Let P,Q,S and T be mappings from a complete metric space (X, d) into itself satisfying the conditions (1.1) and (1.2) Suppose that

One of P,Q,S and T is continuous. (2.1)

Paris S,P and T,Q are compatible on X. (2.2)

Then P,Q,S and T have a unique common fixed point in X.

Proof : Let $\{y_m\}$ be the sequence in X defined by (1.3). By Lemma 1.2 $\{y_m\}$ is a Cauchy sequence and hence it converges to some point u in X. Consequently, the subsequences $\{Sx_{2m}\}$, $\{Px_{2m}\}$, $\{Tx_{2m-1}\}$ and $\{Qx_{2m+1}\}$ also converge to u.

Now suppose that P is continuous. Since S and P are compatible on X, Lemma 1.1 gives that

P^2x_{2m} and $SPx_{2m} \to Pu$ as $m \to \infty$.

By (1.2). we obtain

$$d(SPx_{2m}, Tx_{2m-1}) \leq \alpha \,\frac{d(Qx_{2m-1},Tx_{2m-1})\,[1+d(P^2x_{2m}, SPx_{2m})]}{[1+d(P^2x_{2m}, Qx_{2m-1})]}$$

$$+ \beta\, d\,(P^2x_{2m}, Qx_{2m-1}).$$

By letting $m \to \infty$. we have

$$d(Pu,u) \leq \alpha \,\frac{d(u,u)\,[1+d(Pu,Pu)]}{[1+d(Pu,u)]} + \beta\, d\,(Pu,u)$$

$$\leq \beta\, d\,(Pu, u).$$

which implies

$$(1- \beta\,)\, d\,(Pu, u) \leq 0\,.$$

so that $u = Pu$. By (1.2), we also obtain

$$d(Su,Tx_{2m-1}) \leq \alpha \,\frac{d(Qx_{2m-1},Tx_{2m-1})\,[1+d(Pu,Su)]}{[1+d\,(Pu,Qx_{2m-1})]}$$

$$+ \beta\, d\,(Pu, Qx_{2m-1}).$$

By letting m $\to$ ∞. we have

$$d(Su,u) \leq \alpha \,\frac{d(u,u)\,[1+d(Pu,Su)]}{[1+d(Pu,u)]} + \beta\, d\,(Pu,u)$$

$$\leq \beta\, d\,(Pu, u).$$

so that $u = Su$. Since $S(x) \subset Q(x)$ and hence there exists a point v in x such that $u = Su = Qv$.

$$d(u,Tv) = d(Su.Tv)$$

$$\leq \alpha \quad \frac{d(Qv,Tv)\,[1+d(u,Su)]}{[1+d(u,Qv)]} + \beta\, d\,(u,Qv)$$

which implies that $u = Tv$. Since T and Q are compatible on x and $Qv = Tv = u$, $d(QTv,\ TQv) = \theta$ and hence $Qu = QTv = TQv = Tu$

Moreover, by (1.2), we obtain

$$d(u,Qu) = d(Su,Tu)$$

$$\leq \alpha\ \frac{d(Qu,Tu)\,[1+d(u,Su)]}{[1+d(u,Qu)]} + \beta\, d\,(u,Qu),$$

so that $u = Qu$. Therefore, u is a common fixed point of P,Q,S and T. Similarly, we can also complete the proof, when Q is continuous.

Next suppose that S is continuous. Since S and P are compatible on X, it follows from Lemma (1.1) that

$$S^2x_{2m} \text{ and } PSx_{2m} \rightarrow Su \text{ as } m \rightarrow \infty.$$

By (1.2), we have

$$d(S^2x_{2m}\,,Tx_{2m-1}) \leq \alpha\, \frac{d(Qx_{2m-1}\,,Tx_{2m-1})\,[1+d(PSx_{2m}\ S^2x_{2m})]}{[1+d(PSx_{2m},\ Qx_{2m-1})]}$$

$$+\ \beta\, d\,(PSx_{2m},Qx_{2m-1}).$$

By letting $m \rightarrow \infty$, we obtain

$$d(Su,u) \leq \alpha\ \frac{d(u,u)\,[1+d(Su,Su)]}{[1+d(Su,u)]} + \beta\, d\,(Su,u)$$

i.e. $d(Su,u)\ \leq \beta\, d\,(Su,\ u)$,

which implies

$$(1-\beta)\, d(Su,u) \leq 0,$$

so that $u = Su$. Hence, there exists a point ω in X, such that $u = Su = Q\omega$.

$$d(S^2x_{2m}, T\omega) \leq \alpha \frac{d(Q\omega, T\omega)\,[1+d(PSx_{2m}\ S^2x_{2m})]}{[1+d(PSx_{2m}, Q\omega)]} + \beta\, d\,(PSx_{2m}, Q\omega).$$

By letting $m \to \infty$, we have

$$d(Su,T\omega) \leq \alpha \frac{d(Q\omega,T\omega)\,[1+d(u,Su)]}{[1+d(u,Q\omega)]} + \beta\, d\,(u,Q\omega).$$

which implies that $u = T\omega$. Since T and Q are compatible on X and $Q\omega = T\omega = u$, $d\,(QT\omega, TQ\omega) = 0$ and hence $Qu = QT\omega = Tu$. Moreover by (1.2). we have

$$d(Sx_{2m}, Tu) \leq \alpha \frac{d(Qu,Tu)\,[1+d(Px_{2m}\ Sx_{2m})]}{[1+d(PSx_{2m}, Qx_{2m-1})]} + \beta\, d\,(Px_{2m}, Qu).$$

By letting $m \to \infty$, we obtain

$$d(Su,Tu) \leq \alpha \frac{d(u,Tu)\,[1+d(u,Su)]}{[1+d\,(u,u)]} + \beta\, d\,(u,u),$$

which implies

$$(1-\alpha)\, d(u,Tu) \leq 0\,.$$

so that $u = Tu$. Since $T(x) \subset P(x)$. There exists a point z in x such that $u = Tu = Pz$.

$$d(Sz,u) = d(Sz,\ Tu)$$

$$\leq \alpha \frac{d(Qu,Tu)\,[1+d(Pz,Sz)]}{[1+d\,(Pz,Qu)]} + \beta\, d\,(Pz,Qu)$$

$$\leq \alpha \frac{d(u,u)\ [1+d(Pz,Sz)]}{[1+d\ (Pz,u)]} + \beta\, d\,(Pz,u)$$

$$\leq \beta\,(Pz,u),$$

so that $Sz = u$. Since S and P are compatible on X and $Sz = Pz = u$. $d(PSz,SPz) = 0$ and hence $Pu = SPz = SPz = Su$. Therefore u is a common fixed points of P,Q,S and T. Similarly, we can also complete the proof when T is continuous.

Finally, in order to prove the uniqueness of u, suppose that u and z. u≠z, are common fixed points of P,Q,S and T. Then by (1.2), we obtain

$$d(u,z) = d(Su,Tz)$$

$$\leq \alpha \frac{d(Qz,Tz)\ [1+d(Pu,Su)]}{[1+d\ (Pu,Qz)]} + \beta\, d\,(Pu,Qz)$$

$$d(u,z)\ \leq \alpha \frac{d(z,z)\ [1+d(u,u)]}{[1+d\ (u,z)]} + \beta\, d\,(u,z),$$

$$\leq \beta\, d\,(u,z),$$

which implies

$$(1-\beta)\ d(u,z) \leq 0.$$

which is contraction. Hence $u = z$. This completes the proof.

The following corollary follows immediately from theorem 2.1.

Corollary 2.2. Let P,Q,S and T be mappings from a complete metric space (X,d) into itself satisfying the conditions (1,1), (2.1), (2.2) and

(2.3) $$d(Sx,Ty) \leq \alpha \frac{d(Qy,Sx)\ [1+d\ (Px,Ty)]}{[1+d\ (Px,Qy)]} + \beta\, d\,(Px,Qy)$$

for all x, y in X, where $\alpha, \beta \leq 0$, $\alpha + \beta < 1$. Then P, Q, S and T have a unique common fixed point in X.

Corollary 2.3 : Let P, Q, S and T be mappings from a complete metric space (X,d) into itself satisfying the conditions (1.1) (2.1). (2.2) and

$$d(Sx,Ty) \leq \alpha \frac{d(Px,Ty)\,[1+d\,(Qy,Sx)]}{[1+d\,(Px,Qy)]} + \beta\, d\,(Px,Qy). \qquad (2.4)$$

for all x, y in X, where $\alpha, \beta \geq 0$. $\alpha + \beta < 1$. Then P, Q, S and T and T have a unique common fixed point in X.

Theorem 2.4 : Let P, Q, S and T be mappings from a complete metric space (X,d) into itself satisfying the conditions (2.1), (2.5) and (2.6) for some positive integers s, t, p and q

$$S^s(x) \subset Q^q(x) \text{ and } T^t(x) \subset P^p(x) \qquad (2.5)$$

$$d(S^s x, T^t y) \leq \alpha \frac{d(Q^q y, T^t y)\,[1+d(P^p x, S^s x)]}{[1+d(P^p x, Q^q y)]} + \beta\, d(P^p x, Q^q y) \qquad (2.6)$$

for all x, y in X, where $\alpha, \beta \geq 0$. $\alpha + \beta < 1$. Suppose that

S and T are commuting with P and Q respectively. (2.7)

Then P, Q, S and T have a unique common fixed point in X.

Proof : Since S and T commute with P and Q, S^s and T^t also commute with P^p and Q^q respectively. Thus, by theorem 2.1, there exists a unique point u in X such that $u = P^p u = Q^q u = S^s u = T^t u$.

From this, we obtain $Su = S^s(Su) = P^p(Su)$. Therefore, Su is a common fixed point of S^s and P^p. Also, Tu is a common fixed point of T^t and Q^q. Let $x = Su$ and $y = Tu$ in (2.6). Then we obtain,

$$d(Su,Tu) = d(S^{s}x\ ,\ T^{t}y)$$

$$\leq \alpha\,\frac{d(Q^{q}y,T^{t}y)\ [1+d(P^{p}x,S^{s}x)]}{[1+d(P^{p}x,Q^{q}y)]} + \beta\, d(P^{p}x,Q^{q}y)$$

$$\leq \beta\, d(P^{p}x,Q^{q}y)$$

$$\leq \beta\, d(Su,Tu),$$

which implies that $Su = Tu$ and hence it is the common fixed point of P^{p}, Q^{q}, S^{s} and T^{t}. Further, we obtain $Pu = S^{s}(Pu) = P^{p}(Pu)$. Therefore, Pu is a common fixed point of S^{s} and P^{p}. Also Qu is a common ifxed point of T^{t} and Q^{q}. Let $x = Pu$ and $y = Qu$ in (2.6). then we obtain

$$d(Pu,\ Qu) = d(S^{s}x,\ T^{t}y)$$

$$\leq \alpha\,\frac{d(Q^{q}y,T^{t}y)\ [1+d(P^{p}x,S^{s}x)]}{[1+d(P^{p}x,Q^{q}y)]} + \beta\, d(P^{p}x,Q^{q}y)$$

$$\leq \beta\, d(Pu,Qu),$$

which implies that $Pu = Qu$ and hence it is common fixed point of P^{p}, Q^{q}, S^{s} and T^{t}. By uniqueness u in X, this shows that $u = Pu = Qu = Su = Tu$. This completes the proof.

Remarks : Theorem 2.1 generalizers the result of G. Jungck [7] by using any one continuous mapping as opposed to the continuity of both P and Q.

Theorem 2.1 and corollary 2.2 and 2.3 also generalize the result of B. Fisher [3] by employing compatible mappings instead of commuting mappings. Further the condition (1.2) is more general than the condition of B. Fisher [3].

Theorem 2.4 extends the result of M.S. Khan and M.Imdad [9] by increasing the number of mappings and using any one continuous mapping as opposed to the continuity of both P and Q.

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Journal of Natural & Physical Sciences Vol. 11 (1997) 13-20

A COMMON FIXED POINT THEOREM

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(Received 26.02.1996)

ABSTRACT

The purpose of this paper is to obtain a common fixed point theorem which gives proper generalizations of recent results due to Jachymski; Pant, Joshi and Pande; and Rhoades, Park and Moon. Our reuslt applies to a much wider class of mappings than covered by known results of this type.

Mathematics Subject Classification : 54 H 25.

Keywords : Compatible mappings, fixed point theorems, contractive conditions, coincidence points.

INTRODUCTION

The study of common fixed points of contractive type mappings is presently an area of intense research activity. Most of the recent results deal with four mappings or a sequence of mappings. The most general common fixed point theorems for sequences of mappings are those due to Jachymski [2], Pant, Joshi and Pande [9] and Rhoades, Park and Moon [10]. The theorems concerning sequences of mappings generally require each mappings to satisfy a compatibility condition , a condition on its

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range and a strong type of contractive condition. In the present paper, we obtain a common fixed point theorem concerning a sequence of mappings under much weaker conditions. Thus, our theorem gives a proper generalization of known results in many respects.

Two selfmaps A and S of a metric space (X,d) are called compatible if $\lim_n d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X. This notion was introduced by Jungck [4]. It is well known that compatibility implies commutativity at coincidence points.

Let A_1, A_2, S and T be selfmappings of a set X such that $A_1X \subset TX$ and $A_2X \subset SX$. For x_0 in X, a sequence $\{y_n\}$ defined by $y_{2n} = A_1x_{2n} = Tx_{2n+1}$ and $y_{2n+1} = A_2x_{2n+1} = Sx_{2n+2}$ is called an S,T- iteration of x_0 under A_1 and A_2.

RESULTS

If $\{A_i\}$, $i=1,2,3,\ldots$, S and T be selfmappings of a metric space (X,d), in the sequel we shall denote

$$M_{1i}(x,y) = \max\{d(Sx,Ty), d(A_1x, Sx), d(A_1y, Ty), [d(A_1x, Ty)+d(A_1y,Sx)]/2\}.$$

Theorem : Let $\{A_i\}$, $i=1,2,3,\ldots$, S and T be selfmappings of a metric space (X,d) such that

(i) $A_1X \subset TX$ and $A_2X \subset SX$,

(ii) $d(A_1x,A_2y) \le h\,M_{12}(x,y)$, $0 \le h < 1$,

(iii) $d(A_1x, A_iy) < M_{1i}(x,y)$.

Let S be compatible with A_1 and T be compatible with A_i for some $i > 1$. Suppose that range of one of the mappings A_1, A_2, S or T be a complete subspace of X. Then all the A_i, S and T have a unique common fixed point.

Proof : Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by

$$y_{2n}=A_1x_{2n}=Tx_{2n+1},\ y_{2n+1}=A_2x_{2n+1}=Sx_{2n+2}.$$

This can be done by virtue of (i). Then, by virtue of (ii), we get

$$d(y_{2n},y_{2n+1})\leq h\,d(y_{2n-1},y_{2n})$$

and $d(y_{2n-1},y_{2n})\leq h\,d(y_{2n-2},y_{2n-1})$.

From these inequalities we infer that $\lim_{n\to\infty} d(y_n,y_{n+1})=0$. Also, for any integer $p>0$, we have

$$d(y_{2n+1},y_{2(n+p)+2}\leq h\,d(y_{2n},Y_{2(n+p)+1}+hd(y_{2n},y_{2n+1}).$$

Then $d(y_{2n},y_{2(n+p)+1}\leq d(y_{2n},y_{2n+1})+d(y_{2n+1},Y_{2(n+p)+2})+d(y_{2(n+p)+2.}\,y_{2(n+p)+1})$

$$<hd(y_{2n},Y_{2(n+p)+1})+(2+h)d(y_{2n},y_{2n+1}),$$

that is, $(1-h)\,d(y_{2n},Y_{2(n+p)+1})<(2+h)\,d(y_{2n},y_{2n+1})$.

Similarly, $(1-h)\,d(y_{2n-1},Y_{2(n+p)})<(2+h)\,d(y_{2n},y_{2n-1})$.

Since $\lim_{n\to\infty} d(y_n,y_{n+1})=0$, the above inequalities yield

$$\lim_{n\to\infty} d(y_{2n},y_{2(n+p)+1})=0=\lim_{n\to\infty} d(y_{2n-1},Y_{2(n+p)})$$

Hence $\{y_n\}$ is a Cauchy sequence.

Therefore, by our assumption, $\{y_n\}$ converges to a point in $SX\cup TX$. Let $y_n\to Su$ for some u in X. Then $y_{2n}=A_1x_{2n}=Tx_{2n+1}\to Su$ and $y_{2n+1}=A_2x_{2n+1}=Sx_{2n+2}\to Su$.

Now, using (ii) we get $d(A_1u, A_2x_{2n+1}) \le h\, M_{12}(u, x_{2n+1})$. On letting $n \to \infty$ this inequality yields $d(A_1u, Su) \le h\, d(A_1u, Su)$, that is, $A_1u = Su$. Since the range of A_1 is contained in the range of T, there exists a point w in X such that $A_1u = Tw$. We show that $Tw = A_iw$ for each $i > 1$. If $A_iw \neq Tw$ for some $i > 1$, using (iii) we get

$$d(A_1u, A_iw) < M_{1i}(u, w) = d(A_1u, A_iw),$$

a contradiction. Hence $Su = A_1u = Tw = A_iw$ where $i > 1$.

Let T be compatible with A_j for some $j > 1$. Then, since compatible maps commute at coincidence points, we get $A_jTw = TA_jw$, or equivalently, $A_jA_jw = A_jTw = TA_jw = TTw$. We assert that $A_jw = A_jA_jw = TA_jw$. If $A_jw \neq A_jA_jw$, using (iii) we get

$$d(A_jw, A_jA_jw) = \mathrm{d}(A_1u, A_jA_jw) < M_{1j}(u, A_jw) = d(A_1u, A_jA_jw),$$

a contradiction. Thus $A_jw = A_1u$ is a common fixed point of A_j and T. Similarly, since A_1 and S are compatible and since u is their coincidence point, A_1u is a common fixed point of A_1 and S. Moreover, if $A_1u \neq A_iA_1u$ for some $i > 1$, using (iii) we get

$$d(A_1u, A_iA_1u) = d(A_1u, A_iA_jw) < M_{1i}(u, A_jw) = d(A_1u, A_iA_1u),$$

a contradiction. Thus $A_1u = Su$ is a common fixed point of all the A_i, S and T. The proof is similar when it is assumed that $\{y_n\}$ converges to a point in TX. Uniqueness of the common fixed point follows easily. This complete the proof of the theorem.

The following example illustrates our theorem.

Example : Let $X = [2,6]$ with usual metric d. Define mappings A_i, S, T: $X \to X$, $i = 1, 2 \ldots$, by

$A_1x = 2$ for each x

$A_2x = 2$ if $x \le 3$ or > 4, $A_2x = x - 1$ if $3 < x \le 4$

$Sx = x$ if $x \le 3$, $Sx = 4$ if $x>3$

$Tx = 2$ if $x=2$ or > 4, $Tx=6$ if $2 < x \le 3$, $Tx=3x-7$ if $3 < x \le 4$

$A_3x = 2$ if $x \le 7/3$ or > 3, $A_3x=(19/3)-x$ if $(7/3) < x \le 3$

and for $i > 3$

$A_ix=2$ if $x \le 2+(1/i)$ or > 4, $A_ix = 6+(1/i)- x$ if $2+(1/i) < x \le 3$,

$A_ix = x-1$ if $3 < x \le 4$.

Then $\{A_i\}$, S and T satisfy all the conditions of our theorem and have a unique common fixed point $x = 2$.

It may be observed in the above example that $A_1X \subset TX, A_2X \subset SX$, S is compatible with A_1 and T is compatible with A_3.

In view of the above example, we now compare our theorem with some well known results.

1. Our theorem is more general than Theorem 5.1 of Jachymski [2] in many respects. Jachymski's theorem assumes $A_iX \subset SX$ and A_i to be compatible with T for each $i \ge 2$. These strong conditions are neither required in our theorem nor satisfied in the above example. In our example, it is obvious that A_iX is not contained in SX when $i > 2$. To see that A_2 and T are noncompatible, let us consider a decreasing sequence $\{x_n\}$ in $X= [2,6]$ such that $3 < x_n < 10/3$ and $x_n \to 3$. Then $A_2x_n \to 2$, $Tx_n \to 2$, $A_2Tx_n \to 2$ and $TA_2x_n \to 6$. Hence A_2 and T are noncompatible. Similarly, by considering a decreasing sequence $\{x_n\}$ such that $x_n \to 3$ and $3 < x_n < 3+ (1/3i)$, it can be shown that A_i and T are noncompatible when $i >3$. It is also required in Jachymski's theorem that, for $i \ge 2$, A_1, A_i satisfy the contractive condition $d(A_1x,A_iy) \le f_i(M_{1i}(x,y))$ where $f_i : R_+ \to R_+$ is an upper semicontinuous function such that $f_i(t) < t$ for each $t > 0$. In the above example, for every $i > 2$, A_1 and A_i donot satisfy such a contractive

condition since the required function $f_i(t)$ will not be upper semicontinuous at $t = 2$.

Similarly, our theorem is more general than Theorem 3.3 of Jachymski [2] since that can be obtained as a special case of our theorem. However, in the above example, no pair of mappings A_1, A_i, $i \geq 2$, satisfies the conditions of Theorem 3.3 of [2].

II. Our theorem is more general than that due to Rhoades, Park and Moon [10] in following respects. The theorem of Rhoades et al requires $A_iX \subset SX \cap TX$ and A_i to be compatible with both S and T for each value of i. Our theorem assumes much weaker conditions than these. Moreover, their result requires each A_i, A_j to satisfy an $(\in, h)$ - contractive condition (a suitable correction to their $(\in, h)$ condition appeared in Jungck et al. [5]), h being lower semicontinuous. However, in the above example, A_1, A_i donot satisfy such a contractive condition for $i > 2$ since h will not be lower semicontinuous at $\in = 2$.

III. Likewise, the present theorem is more general than Theorem 2 of Pant, Joshi and Pande [9] in two respects. In Theorem 2 of Pant et al if we take $\{P_i\} = \{A_1, A_1, A_1, \ldots.\}$ and $\{Q_i\} = \{A_2, A_3, \ldots.\}$ then A_1, A_i are required to satisfy an $(\in, h)$ - contractive condition for each $i > 2$, h being nondecreasing. This condition is not satisfied in the above example since h is not found to be nondecreasing. Secondly, in Theorem 2 of Pant et al T is assumed to be compatible with a particular mapping while in the present theorem T may be compatible with any A_i for $i > 1$.

It is clear from the above discussion that our theorem applies to a much wider class of mappings than covered by the above mentioned results in [2], [9], [10]. Moreover, since other results of this type can be obtained as special cases of the results in [2], [9], [10], they can also be obtained as special cases of our theorem. Among the results that can be obtained as special cases of our theorem or can be generalized in the light of our theorem we mention those due to Fisher [1], Joshi and Pant [3], Meir and Keeler [6], Pant [7], [8], Sessa [11] and Tivari and Singh [12].

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Journal of Natural & Physical Sciences Vol. 11 (1997) 21-38

ON KANAN TYPE MAPS IN D-METRIC SPACES

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(Received 5.3.1996)

ABSTRACT

In this paper some fixed point theorems for the mappings in D-metric spaces satisfying certain contractive conditions are proved which include some well-known fixed point theorems of Banach [1], Kannan [10] and Maia [11] in ordinary metric spaces, and of Dhage [3,8] in D-metric spaces as the special cases.

Mathematics Subject Classifications : 47H10, 54H25

Keywords : D-metric space, contraction map, fixed point etc.

INTRODUCTION

The fixed point theory is one of the important topics that has been discussed in topological spaces and which has several important applications in the nonlinear analysis. In case of a metric space, the fixed point theory is culminated to its peak point and at present a considerable literature is available in this direction. In almost all fixed point theorems in metric spaces a sufficient condition is given in terms of contractive inequality which guarantees the existence of fixed point of the mappings in question. In this paper some sufficient conditions, like in ordinary metric

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spaces, are given to the mapings in D-metric spaces which guarantee the existence of the fixed point.

Recently the present author [2,3,4] introduced the notion of a D-metric space as follows.

Let X denote a non-empty set. A real function D on $X xXx X$ is said to be a D-metric on X if it satisfies the following properties :

(M_1) $D(x,y,z) \geq 0$ for all $x,y,z \in X$, (nonnegetivity)

(M_2) $D(x,y,z) = 0$ if and only if $x=y=z$, (coincidence)

(M_3) $D(x,y,z) = D(x,y,z) = \ldots.$ (symmetry)

(M_4) $D(x,y,z) \leq D(x,y,z)+(a,y,z)$

for all $x,y,z \in X$ (tetrahedral inequality).

The non-empty set X together with a D-metric D on X is called a D-metric space and it is denoted by (X,D). The details of a D-metric space and its generalizations upto n variables are given in Dhage [4]. Below we give some basic definitions which will be used in the sequel.

A sequence $\{x_n\}$ of points of a D-metric space X is called D-convergent or simply convergent and converges to a point x if $\lim_{m,n\to\infty} D(x_m,x_n,x)=0$. A Sequence $\{x_n\}$ of points of a D-metric space X is said to be *D-Cauchy* if $\lim_{m,n,p\to\infty} D(x_m,x_n,x_p)=0$. A *complete* D-metric space X is one in which every *D-Cauchy* sequence converges to a point in it. A D-metric space X is said to be bounded if there exists a constant $M>0$ such that $D(x,y,z) \leq M$ for all $x,y,z \in X$, and the constant M is called a *D-bound* of X.

It is shown in Dhage [4] that the D-metric D is a continuous function in the topology of D-metric convergence which is Hausdorff in nature.

Let $f : X \rightarrow X$, then an orbit of f at a point $x \in X$ is a set $O(x)$ in X given by

$$O(x) = \{x, fx, f^2x, \ldots\} \tag{1.1}$$

An orbit $O(x), x \in X$, of f in a D-metric space X is said to be bounded if there exists a constant K such that $D(u,v,w) \leq K$ for all $u,v,w \in O(x)$.

A D-metric space is called f-orbitally bounded if the for each $x \in X$, $O(x)$is bounded. It is clear that every bounded D-metric space is f-orbitally bounded, but the converse may not be true.

Again a D-metric space X is said to be *f-orbitally complete* if every D-Cauchy sequence $\{x_n\}$ in $O(x), x \in X$, converges to a point in X. Finally a mapping f: $X \rightarrow X$ is said to be *f-orbitally continuous* if $x_n \rightarrow u$, $\{x_n\} \subset O(x)$, $x \in X$, then $fx_n \rightarrow fu$

Kannan [10] proved the following interesting fixed point theorem in the ordinary metric spaces which is responsible for the huge development of the metric fixed point theory.

Theorem 1.1 : Let the mapping f on a complete metric space into iteself satisfy

$$d(fx,fy) \leq \alpha[d(x,f(x)+d(y,fy)] \tag{1.2}$$

for all $x, y \in X$ and $0 \leq \alpha < 1/2$. Then f has a unique fixed point.

The mappings satisfying (1.2) are called Kannan mappings and are different from well-known Banach mappings characterized by the inequality

$$d(fx,fy) \leq \alpha\, d(x,y) \tag{1.3}$$

for all $x, y \in X$ and $0 \leq \alpha < 1$.

In this paper we investigate some new classes of contractive mappings in D-metric spaces possessing the fixed point property. In the following section we prove the main results of this paper.

MAIN RESULTS

Before going to the main results we prove a useful lemma.

Lemma 2.1 : Let $\{x_n\}$ be a bounded sequence in a D-metric space X with D-bound K satisfying

$$D(x_n, x_{n+1}, x_m) \leq \alpha^n K \tag{2.1}$$

for all $m>n \in N$, where $0 \leq \alpha < 1$. Then $\{x_n\}$ is D-Cauchy.

Proof : From (2.1), it follows that

$$D(x_n, x_{n+1}, x_{n+p}) \leq \alpha^n K$$

and $D(x_n, x_{n+1}, x_{n+p+t}) \leq \alpha^n K$ (2.2)

for all $p, t \in N$.

Now

$$\begin{aligned} & D(x_n, x_{n+p}, x_{n+p+t}) \\ & \leq D(x_n, x_{n+1}, x_{n+p+t}) + D(x_n, x_{n+p}, x_{n+p+t}) \\ & \qquad + D(x_n, x_{n+p}, x_{n+p+t}) \\ & \leq D(x_n, x_{n+1}, x_{n+p}) + D(x_n, x_{n+1}, x_{n+p+t}) \\ & + D(x_{n+1}, x_{n+2}, x_{n+p}) + D(x_{n+1}, x_{n+2}, x_{n+p+t}) \\ & + \ldots\ldots \\ & \leq \sum_{k=1}^{n+p-1} \alpha^k (K) + \sum_{k=1}^{n+p+t-1} \alpha^k (K) \end{aligned}$$

$$(\because D(x_i, x_j, x_k,) \leq K, \forall\, i, j, k)$$

$$\leq 2\left(\sum_{k=1}^{n+p+t} \alpha^k\right)K$$

$$= \frac{2\ \alpha^n}{1-\ \alpha}(K)$$

$\to 0$ as $n \to \infty$.

This shows that $\{x_n\}$ is D-Cauchy and the proof of the lemma is complete.

As a direct consequences of Lemma 2.1, we obtain

Lemma 2.2 : let X be a f-orbitally bounded D-metric space satisfying

$$D(f^n x, f^{n+1} x, f^m x) \leq \alpha^n K \tag{2.3}$$

for all $m>n \in N$ and $x \in X$, where $0 \leq \alpha < 1$, and K is the D-bound of $O(x)$ in X. Then $\{f^n x\}$ is D-Cauchy for all $x \in X$.

Theorem 2.1 : Let f be a self - map of a f-orbitally bounded and f-orbitally complete D-metric space X satisfying

$$D(fx,fy,fz) \leq \alpha \max \{D(x,y,z), D(x,fx,fy), D(y,fy,fz), D(z,fz,fx) \\ D(x,fx,z), D(y,fy,x), D(z,fz,y)\} \tag{2.4}$$

for all $x,y,z \in X$, and $0 \leq \alpha < 1$. then f has a unique fixed point.

Proof : Suppose that $x=x_0 \in X$ is an arbitrary point and consider the sequence $\{x_n\}$ in defined by

$$x_{n+1}=fx_{n,}\ n=0,1,2,..... \tag{2.5}$$

If $x_r=x_{r+1}$ for some $r \in N$, then $u = x_r$ is fixed point of f. Therefore, we assume that $x_n \neq x_{n+1}$ for each $n=0,1,2.$ ... We show that $\{x_n)$ is D-Cauchy sequence. Let $x = x_{n-1}$, $y = x_n$ and $z = x_{n+1}$, then by inequality (2.4), we obtain

$$D(x_n,x_{n+1},x_{n+2})$$

$$=D(fx_{n-1},f(x_n,fx_{n+1})$$

$$\leq \alpha \max \{D(x_{n-1},x_n,x_{n+1}), D(x_n,x_{n+1},x_{n+2})\}$$

Since $D(x_n,x_{n+1},x_{n+2}) \leq \alpha D(x_n,x_{n+1},x_{n+2})$, $\alpha < 1$, is not possible, we have

$$D(x_n,x_{n+1},x_{n+2}), \leq \alpha D(x_{n-1},x_n,x_{n+1}),$$

for all $n \in N$.

Therefore

$$D(x_n,x_{n+1},x_{n+2}) \leq \alpha D(x_{n-1},x_n,x_{n+1})$$

$$\leq \alpha^n D(x_0,x_1,x_2) \qquad (*)$$

Now for $m>1$, we have

$$D(x_1,x_2,x_m)$$

$$=D(fx_0,fx_1,fx_{m-1})$$

$$\leq \alpha \max \{D(x_0,x_1,x_2), D(x_0,x_1,x_{m-1}), D(x_2,x_m,x_{m-1})\}$$

$$\leq \alpha \max \{\alpha^0 K, \alpha^0 \max D(x_a,x_b,x_{m-1})\}$$

$$0 \leq a \leq m-1$$

$$1 \leq b \leq m$$

$$\leq \alpha K.$$

Similarly for $m>2$, we have

$$D(x_2,x_3,x_m)$$

$$= D(fx_1,fx_2,fx_{m-1})$$

$$= \alpha \max \{D(x_1,x_2,x_{m-1}), D(x_1,x_2,x_3), D(x_{m-1},x_m,x_2)\}$$

$$\leq \alpha \max \{\alpha K, D(x_{m-1},x_m,x_2)\} \qquad (**)$$

But

$$
\begin{aligned}
D(x_{m-1},x_m,x_2) &= D(x_m,x_2,x_{m-1})\\
&= D(fx_{m-1},fx_1,fx_{m-2})\\
&= \alpha\max\{D(x_1,x_{m-1},x_{m-2}), D(x_1,x_2,x_{m-1}),D(x_2,x_{m-1},x_{m-2})\}
\end{aligned}
$$

Substituting this in (**), we obtain

$$
D(x_2,x_3,x_{\mathrm{m}}) \le \alpha \max\{\alpha K,\ \alpha \max_{\substack{0\le a\le m-1\\ 1\le b\le m}} D(x_{a'},x_{b'},x_{m-2})\}
$$

$$
\le \alpha^2 K
$$

In general

$$
\begin{aligned}
D(x_n,x_{n+1},x_m) &= D(fx_{n-1},fx_n,fx_{m-1})\\
&\le \alpha \max\{\alpha^{n-1} K,\ \alpha^{n-1} \max_{\substack{0\le a\le m-1\\ 1\le b\le m}} D(x_{a'},x_{b'},x_{m-n})\}\\
&\le \alpha^{n-1} K
\end{aligned}
$$

for all $m > n \in N$.

Now an application of Lemma 2.2 yields that $\{x_n\}$ is a D-Cauchy sequence in X. Since X is a complete D-metric space, there is a point, say $u \in X$ such that $\lim D(x_m, x_n, u)=0$, i.e. $x_n \rightarrow u$. But every D-metric space X is hausdroff in the topology of D-metric convergence and so such a point u is unique, see for details Dhage [4]. Now we show that u is a fixed point of f.

If $u \neq fu$, then $D(u,u,fu)\neq 0$, $D(u,fu,fu)\neq 0$ and

$$\begin{aligned}
D(u,u,fu) &= \lim_{n\longrightarrow \infty} D(x_{n+1},x_{n+1},fu)\\
&= \lim_{n\longrightarrow \infty} D(fx_n,fx_n,fu)\\
&\leq \lim_{n\longrightarrow \infty} \max\{D(x_n,x_n,u),\, D(x_n,x_{n+1},x_{n+2}),\\
&\qquad D(x_n,x_{n+1},u),\, D(u,fu,x_{n+1}),\\
&\qquad D(x_n,x_{n+1},u),\, D(u,fu,x_n)\}\\
&= \max\{0, D(u,u,fu)\}\\
&= \alpha D(u,u,fu)
\end{aligned}$$

which is a contradtiction, since $\alpha < 1$. Hence $u = fu$, i.e. u is a fixed point of f.

To prove the uniquencess, let $v(\neq u)$ be another fixed point of f. Then by condition (2,4) we get

$$\begin{aligned}
D(u,u,v) &= D(fu,fu,fv)\\
&\leq \alpha \max\{D(u,u,v), D(u,u,v)\}\\
&= \alpha D(u,v,v) \qquad (2.6)
\end{aligned}$$

since $D(u,v,v) \leq \alpha D(u,v,v)$, $\alpha < 1$, is not possible.

Similarly by interchanging the role of u and v it is shown that

$$D(u,v,v) \leq \alpha\, D(u,u,v). \tag{2.7}$$

From (2.6) and (2.7), it follows that $D(u,v,u) \leq \alpha\, D(u,u,v)$, $\alpha < 1$, which is a contradiction and hence $u=v$. Thus f has *a* unique fixed point and the proof of the theorem is complete.

As a consequence of Theorem 2.1 we obtain the following corollaries.

Corollary 2.1 : Let f be a self-mapping of an f-orbitally bounded and f-orbitally complete D-metric space X. If there exists a $p \in N$ such that

$$\begin{aligned} &D(f^p x, f^p y, f^p z) \\ &\qquad \leq \alpha \max\,(D(x,y,z)\ D(x, f^p x, f^p y)\ D(y, f^p y, f^p z), D(z, f^p z, f^p x) \\ &\qquad\qquad D(x, f^p x, z)\ D(y, f^p y, x), D(x, f^p z, y)\} \end{aligned} \tag{2.8}$$

for all $x, y, z \in X$ and $0 \leq \alpha < 1$. Then f has a unique fixed point

Corollary 2.2 : (Dhage [2.3]) : Let f be a self-map of an f-orbitally bounded and f-orbitally complete D-metric space X satisfying

$$D(fx,fy,fz) \leq \alpha\, D(x,y,z) \tag{2.9}$$

for all $x,y,z \in X$ and $0 \leq \alpha < 1$. Then f has a unique fixed point

Corollary 2.3 : (See [9,12]) : Let f be a self-map of a complete metric space X satisfying

$$d(fx,fy) \leq \alpha \max\,\{d(x,y)\ d(x,fx), d(y,fy)\} \tag{2.10}$$

for all $x, y, z \in X$ and $0 \leq \alpha < 1$. Then f has a unique fixed point

Proof : Define a D-metric D on X by

$$D(x,y,z)=\max\{d(x,y)\ d(y,z), d(x,z)\} \tag{2.11}$$

for all $x, y, z \in X$ (See Dhage [3]).

Obviously the completeness of (X,d) implies the completeness and hence an f-orbitally completeness of the D-metric space (X,D), Next we show that the completness of (X,d) together with the condition (2.10) implies the f-orbital boundeness of (X,D). Now for any $x \in X$, consider a sequence $\{x_n\}$ of points of X defined by (2.5) Clearly the sequence $\{x_n\}$ is same as the orbit (x) of f at a point $x \in X$. Then for $x=x_{n-1}$ and $y=x_n$, condition (2.10), we get

$$d(x_n,x_{n+1}) \leq \alpha\, d(x_{n-1},x_n) \tag{2.12}$$

for all $n \in N$. The inequality (2.12) shows that the sequence $\{x_n\}$ is Cauchy, and since (X,d) is complete, it converges to a point in x. But we know that every convergent sequence is bounded so $\{x_n\}$ is bounded w.r.t.D. In view of (2.11), it follow that $\{x_n\}$ is bounded w.r.t.D Hence (X,D) is an f-orbitally bounded D-metric space. Finally for any $x,y,z \in X$, from (2.10) and (2.11), we have

$$\begin{aligned} D(fx,fy,fz) & \\ &= \max\{d(fx,fy), d(fy,fz), d(fz,fx)\} \\ &\leq \alpha \max\{d(x,y), d(y,z), d(z,x), d(x,fx)\, d(y,fy), d(z,fz)\} \\ &\leq \alpha \max\{D(x,y,z), D(fx,fy), D(y,fy,fz), D(z,fz,fx)\} \\ &\qquad D(x,fx,x), D(y,fy,x), D(z,fz,y)\} \end{aligned}$$

Thus all the conditions of Theorem 2.1 are satisfied and hence an application of it yields that f has a unique fixed point. This completes the proof.

Remark 2.1 : Since the class of mappings f given by (2.10) includes the class of Kannan's mappings given by (1.2), Theorem 1.1 follows directly from our Theorem 2.2 as the special case.

Maia [11] initiated the study of the fixed point theorems in a metric space with two metrics. Sometimes it is possible that a D-metric space may not be complete w.r.t. a D-metric, but is complete w.r.t. another D-metric. Below in the following we deal with such situation and prove some fixed point theorems in tri-metric spaces which include some well-known fixed point theorems in ordinary metric spaces on the lines of Maia [11] as the special cases.

Theorem 2.2 : Let X be a D-metric space with three D-metrics D, D_1 and D_2 and $f : X \to X$. Assume the following conditions hold in X.

(i) $D_2(x,y,z) \leq D_1(x,y,z) \leq D(x,y,z)$ for all $x,u{=}y,z \in X$,

(ii) X is f-orbitally bounded w.r.t.D.,

(iii) X is f-orbitally complete w.r.t.D_1.

(iv) f is satisfies (2.4) on X w.r.t. D.

(v) f is satisfies (2.4) on X w.r.t.D.

Then f has a unique fixed point.

Proof : Let $x = x_0 \in X$ be arbitrary and consider the sequence $\{x_n\}$ in X defined by (2.5). Then proceeding as in the proof of Theorem 2.1 with similar argument, it is shown that $\{x_n\}$ is a D-Cauchy sequence w.r.t. D.

In view of hypothesis (i), it follows that (X_n) is also D-Cauchy w.r.t.D_1 Since X is f-orbitally complete w.r.t.D_1, there is a point $u \in X$ such that

$$\lim_{m,n\to\infty} D_1(x_m, x_n, u) = 0.$$

Now $D_2, \le D_1$, on X^3, therefore we get

$$\lim_{m,n\to\infty} D_2(x_m, x_n, u) \le \lim_{m,n\to\infty} D_1(x_m, x_n, u) = 0.$$

Again the uniquencess of the fixed point u directly follows from condition (v). This completes the proof.

Corollary 2.4 : Let X be a D-metric space with three D-metrics D, D_1 and D_2 and $f : X \to X$. Assume all the conditions (i) - (v) of Theorem 2.2 hold with (v) replaced by (2.8). Then f has a unique fixed point.

Corollary 2.5 : Let X be a tri D-metric space with three D-metrics D, D_1 and D_2 and $f : X \to X$. Assume all the conditions (i) - (v) of Theorem 2.2 hold with (v) replaced by (2.9). Then f has a unique fixed point.

Corollary 2.6 : Let X be a tri metric space with metrics D, D_1 and D_2 and let $f : X \to X$. Assume all the following conditions hold in X.

(i) $d_2(x,y) \le d_1(x,y) \le d(x,y)$ for all $x, y \in X$,

(ii) X is f-orbitally complete w.r.t. d_1,

(iii) f is f-orbitally continuous w.r.t.d_2, and

(iv) f is satisfies (2.10) on X.

Then f has a unique fixed point.

Proof : Define the D-metrics D, D_1 and D_2 on X by

$$\left.\begin{aligned} D(x,y,z) &= \max\,\{d(x,y)\; d(y,z),\, D(z,x)\} \\ D_1\,(x,y,z) &= \max\,\{d_1(x,y)\; d_1(y,z),\, D_1(z,x)\} \\ \text{and}\quad D_2\,(x,y,z) &= \max\,\{d_2(x,y)\; d_2(y,z),\, D(z,x)\} \end{aligned}\right\} \qquad (2.13)$$

for $x,y,z \in X$. Then proceeding as in the proof of Corollary 2.3 with similar arguments it is shown that all the conditions of Theorem 2.2 are satisfied and hence an application of it yields the desired result. This completes the proof.

Corollary 2.7 : (Maia [11]) : Let X be a bi-metric space with two metrics d and δ and let $f : X \rightarrow X$. Assume all the following conditions hold in X.

(i) $\delta\,(x,y) \leq d(x,y)$ for all $x,y \in X$,

(ii) X is complete w.r.t. δ,

(iii) f is continuous w.r.t. δ. and

(iv) f is satisfies (1.3) on X w.r.t.D.

Then f has a unique fixed point.

Proof : The proof follows directly from Corollary 2.6 by letting $d_1 = d_2 = \delta$.

Finally we deal with a special class of mappings called contractive mappings on a D-metric space which is not necessarily complete and bounded and prove some results on fixed point.

Theorem 2.3 : Let f be a self-map of a D-metric space X satisfying

$$\begin{aligned} D(fx,fy,fz) < \max\{ & D(x,y,z), D(x,fx,fy), D(y,fy,fz), D(z,fz,fx) \\ & D(x,fx,z), D(y,fy,x) D(z,fz,y)\} \end{aligned} \tag{2.14}$$

for all $x,y,z \in X$ with $\max\{D(x,y,z), D(x,fx,fy), D(y,fy,fz), D(z,fz,fx), D(f,fx,z)\, D(y,fy,x)\, D(z,fz,y)\}$ ‡ 0. Suppose further that the sequence $\{x_n\}$ in X defined by (2.5) has a convergent subsequence converging to a point $u \in X$. If f, f^2 and f^3 are continuous at u, then u is a unique fixed point of f.

Proof : Now for $x = x_0$, $y = x_1$ and $z = x_2$ from (2.14) we obtain

$$D(x_1,x_2,x_3) < D(x_0,x_1,x_2).$$

Proceeding in this way, by induction, we get

$$D(x_{n-1}, x_n, x_{n+1}) > D(x_n, x_{n+2}, x_{n+2}) \tag{2.15}$$

for each n, n-1,2,.....

Let $c_n = D(x_n,x_{n+1},x_{n+2})$, then from (2.15), we have

$$c_0 > c_1 > c_2 > \ldots. > c_n > \ldots. \tag{2.16}$$

This shows that $\{c_n\}$ is a strictly decreasing sequence of positive real numbers which is bounded and hence congergent. Thus there is a point c such that

$$\lim_{m,n\to\infty} D_2(x_n, x_{n+1}, x_{n+2}) = \lim_{n\to\infty} c_n = c.$$

Further every subsequence $(c_n\}_k$ of $\{c_n\}$ is strictly decreasing and converges tot he same limit point c.

In view of $c_{n+1} > c_n$ for all $n \in N$ it follows that

$$c_{n_k+1} < c_{n_k} \quad \text{for all } k \in N.$$

Therefore

$$\lim_{k\to\infty} c_{n_k+1} = c = \lim_{k\to\infty} c_{n_k} \tag{2.18}$$

Now suppose that the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to the point u, i.e. $\lim\limits_{k\to\infty} x_{n_k} = u$. By continuity of f (see Dhage (7) we obtain

$$\lim_{k\to\infty} x_{n_k+1} = \lim_{k\to\infty} f(x_{n_k}) = f(\lim_{k\to\infty} x_{n_k}) = fu.$$

Similarly

$$\lim_{k\to\infty} x_{n_k+2} = f^2u \text{ and } \lim_{k\to\infty} x_{n_k+3} = f^3u.$$

Since the D-metric D is continuous, we have

$$\lim_{k\to\infty} D(x_{n_k}, x_{n_k+1}, x_{n_k+2}) = D(u,fu,f^2u)$$

and $$\lim_{k\to\infty} D(x_{n_k+1}, x_{n_k+2}, x_{n_k+3}) = D(fu,f^2u,f^3u).$$

But in view of (2.18), we get

$$D(fu, f^2u, f^3u) = D(u, fu, f^2u). \tag{2.19}$$

If $u \neq fu$, then $D(u, fu, f^2u) \neq 0$ and from (2.14) we obtain

$$D(fu, f^2u, f^3u) < D(u, fu, f^2u) \tag{2.20}$$

which is a contradiction to (2.19). Hence $u = fu$. Again the uniqueness of u follows from the condition (2.14). This completes the proof.

As a consequence of Theorem 2.3 we obtain the following corollaries.

Corollary 2.8 : (Dhage [3]) : Let f be a continuous self-map of a compact D-metric space X satisfying (2.14), then f has a unique fixed point.

Proof : Since every sequence in the compact D-metric space has a convergent subsequence, the proof follows by an application of Theorem 2.3.

Corollary 2.9 : (Dhage [8]) : Let f be a self-map of a D-metric space X satisfying

$$D(fx, fy, fx) < D(x, y, z) \tag{2.21}$$

for all $x, y, z, \in X$ with $D(x, y, z) \neq 0$. Further suppose that the sequence $\{x_n\}$ in X defined by (2.5) has a convergent subsequence congerging to a point $u \in X$. Then u is a unique fixed point of f.

Proof : Since the condition (2.21) implies the continuity of f on X (See Dhage [6]). The conclusion now follows by an application of Theorem 2.3.

Corollary 2.10 : Let f be a self-map of a metric space X satisfying

$$d(fx,fy) < \max\{d(x,y), d(x,fx), d(y,fy)\} \qquad (2.22)$$

for all $x,y \in X$ with $\max\{d(x,y), d(f,fx), d(y,fy)\} \neq 0$. Suppose the sequence $\{x_n\}$ in X defined by (2.5) has a congergent subsequence converging to a point $u \in X$. Further, if f, f^2 and f^3 are continuous at u, then u is a unique fixed point of f.

Proof : Define a D-metric D on X by (2.11). Then it is clear that condition (2.22) implies the condition (2.14). Also the convergence of the subsequence w.r.t. the metric d implies the convergence w.r.t. the D-metric D. It is shown in (Dhage [7]) that the continuity of a mapping at a point w.r.t. D implies the continuity of the mapping at the same point w.r.t. the D-metric D. Hence f, f^2 and f^3 are continuous at $u \in X$ w.r.t. the D-metric on X. Thus all the conditions of Theorem 2.3 are satisfied and so an application of it yields the desired result. The proof is complete.

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Journal of Natural & Physical Sciences Vol. 11 (1997) 39-46

BALANCED ROW COLUMN DESIGNS

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(Received 30.5.96)

ABSTRACT

In this paper balanced designs eliminating heterogeneity in two directions with number of treatments less than the number of rows and/or columns have been studied.

Mathematics Subject Classification (1991) : 62K99

Keywords : Elementary treatment contrasts, orthogonality, incidence matrices, g-inverse, doubly centered matrix, pseudo-dispersion matrix, Youden square, normal equation.

INTRODUCTION

The simplest form of designs eliminating heterogeneity in two directions are the Latin squares. The use of Latin square is limited by the fact that the number of experimental units needed to conduct this with s treatments is s^2. which increases very rapidly as s increases. When the number of treatments are not euqal to the number of rows and that of the number of columns then a Latin square design is not feasible to plan. An important class of designs

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for such situations was introduced by Youden [8], known as Youden square designs, which are some what less restrictive than the Latin square designs. Some other alternatives to Latin Square designs were suggested by Shrikhande [7], Hoblyn *et.al* [4], Freeman [1,2] and Freeman [3]. In all these designs, the row versus column classification is orthogonal, while the row versus treatment or column versus treatment classification may be non-orthogonal.

Row-column designs have been classified as $O:PP$, $O:TT$, $O:TP$ etc. by Hoblyn *et al.* [4], where O stands for orthogonality, T for balance and P for partial balance. Since the major interest is in the treatment effects, classification of a design on the basis of the overall design rather than on the basis of component designs (row-wise or column-wise design) is more meaningful. In view of this, in this paper, we study row-column designs which are incomplete in row and/or column and are balanced for estimates of elementary treatment contrasts. Unfortunately, it has not been possible to obtain a method for constructing balanced row-column designs with number of treatments less than the number of rows and/or columns.

BALANCED ROW-COLUMN DESIGNS

Let us consider a row-column design in which $n=pq$ experimental units are arranged in p rows and q columns and v treatments are assigned to these pq units in such a manner that each treatment is replicated r times in the design. Let the i-th treatment appears in the j-th row n_{1ij} times and in the k-th column n_{2ik} times where $n_{1ij} = 0$, or 1 and $n_{2ik} = 0$ or 1. We further assume that $p \le v$ and $q \le v$. Let $N_1=((n_{1ij}))$ and $N_2=((n_{2ik}))$ denote the treatments versus rows and treatments versus columns incidence matrices respectively.

It is then known that under usual homoscedastic fixed effect model, the reduced normal equations for estimating treatment effects after eliminating the row and column effects come out as,

$$F\tau = Q$$

where $F = rI - N_1N_1'/q - N_2N_2'/p + (r/v)JJ'$ (2.1)

τ is the vector of treatment effects, Q is the column vector of adjusted treatment totals given by

$$Q = T - N_1R/q - N_2C/p + N_1JG/pq$$

T, R and C being the column vectors of treatment, row and column totals respectively. G is the grand total of the observations, I denotes the identity matrix of order v and $J=(1,\ldots,1)'$. The rank of F is utmost $(v-1)$. We now prove the following results.

Theorem 2.1 : A design eliminating heterogeneity in two directions is balanced if and only if the F matrix of the design is given by,

$$F = \theta\,(I - v^{-1}\,JJ'),$$

where θ is some positive scalar.

Proof : A square matrix A or order n is called a doubly centered matrix if its row sums and column sums are zero (see e.g. Rao and Mitra [6] pp 181). If A is doubly centered of order n and rank (n-1), then the unique doubly-centered generalized inverse (g-inverse) of A is A^+, the Moore-Penrose inverse of A (see Rao and Mitra [6] pp 181-182).

It can easily be seen that F is a doubly centered matrix of order v. Since rank $(F) = v-1$, it follows that a solution of normal equation is

$$\hat{\tau} = F^+Q$$

where F^+ is More-Penrose inverse of F, Further F^+ is also doubly-centered.

The pseudo dispersion matrix of τ is given by σ^2 F^+ where σ^2 is per observation variance. Using the fact that F^+ is also doubly centered, one can easily show that the average variance of all estimated elementary treatment contrasts is given by

$$Aver.\ Var = 2\sigma^2/H$$

where H is the harmonic mean of non-zero eigen values of F.

Let $v^*(\hat{\tau})$ $Cov^*(\hat{\tau}_i, \hat{\tau}_j)$ denote the pseudo-variance of estimated i-th treatment effect and the pseudo-covariance between the estimated i-th and j-th treatment effects respectively. Then,

$$\sum_{i=1}^{v} v^*(\hat{\tau}_i) = tr\ (F^+) = \sigma^2\ (v-1)/H \qquad (2.2)$$

where tr stands for the trace of a matrix. Also,

$$var(\hat{\tau}_1 - \hat{\tau}_2) + Var\ (\hat{\tau}_1 - \hat{\tau}_3) + \ldots Var(\hat{\tau}_1 - \hat{\tau}_v) = v.\,v^*(\hat{\tau}_1) + \sum_{i=1}^{v} v^*(\hat{\tau}_1) \qquad (2.3)$$

and

$$var(\hat{\tau}_2 - \hat{\tau}_1) + Var\ (\hat{\tau}_2 - \hat{\tau}_3) + \ldots Var(\hat{\tau}_2 - \hat{\tau}_v) = v.\,v^*(\hat{\tau}_2) + \sum_{i=1}^{v} v^*(\hat{\tau}_i) \qquad (2.4)$$

The solutions (2.3) and (2.4) follow from the fact that F^+ is doubly-centered.

Now, let the design be balanced, i.e. $Var\ (\hat{\tau}_i - \hat{\tau}_j)$ is a constant independent of i and j. Then it follows from (2.3) and (2.4) that.

$$v^*(\hat{\tau}_1) = v^*(\hat{\tau}_2)$$

Thus, from (2.2) we have,

$$v^*(\hat{\tau}_i) = \sigma^2 (1 - v^{-1})/H \text{ for } i = 1, 2 \ldots, v.$$

Also, since the design is balanced,

$$Var(\hat{\tau}_i - \hat{\tau}_j) = 2\,\sigma^2/H$$

or, $v^*(\hat{\tau}_i) + v^*(\hat{\tau}_j) - 2\,Cov^*\,(\hat{\tau}_i, \hat{\tau}_j) = 2\,\sigma^2/H$, giving,

$$Cov^*(\hat{\tau}_i, \hat{\tau}_j) = -\,\sigma^2/vH, \text{ for } i = j = 1, 2, \ldots, v.$$

Thus, if the design is balanced

$$F^+ = H^{-1}\,(I - v^{-1}JJ')$$

Since $(I - v^{-1}JJ')$ is symmetric and idempotent, it follows that

$$F = H\,(I - v^{-1}\,JJ')$$

conversely, let $F = \theta\,(I - v^{-1}JJ')$

Then,

$$F^+ = \theta^{-1}\,(I - v^{-1}JJ')$$

Thus, $v^*(\hat{\tau}_i) = \sigma^2\,\alpha$ for $i = 1, \ldots, v$, and

$$Cov^*\,(\hat{\tau}_i, \hat{\tau}_j) = \sigma^2\,\beta \text{ for } i \neq j = 1, 2 \ldots v$$

where α and β are constants that do not depend on i and j. Thus,

$$Var\,(\hat{\tau}_i - \hat{\tau}_j) = v^*\,(\hat{\tau}_i) + v^*\,(\hat{\tau}_j) - 2\,Cov^*\,(\hat{\tau}_i, \hat{\tau}_j)$$

is also a constant independent of i and j showing that the design is balanced. Hence the theorem. The result contained in the theorem is well known, but its proof as given above appears to be new.

Theorem 2.2 : A row-column design with v treatments, p rows, q columns and equal replication r of treatment is balanced if, and only if.

$$pN_1N_1' + qN_2N_2' = [r(p+q) - \mu\, I] + \mu\, JJ' \qquad (2.5)$$

where $\mu\,(v\text{-}1) = r\,(2pq\text{-}p\text{-}q)$

Proof : If (2.5) holds, we have,

$$F = rI - N_1N_1'/q - N_2N_2'/p + (r/v)\,JJ'$$

$$= rI - (1/pq)[\{\, r(p+q) - \mu\}\, I + \mu\, JJ'] + (r/v)\,JJ'$$

Since $vr = pq$ and $\mu\,(v\text{-}1) = r\,(2pq\text{-}p\text{-}q)$. Thus,

$$F = (\mu - r^2)n^{-1}\,(I - v^{-1}\,JJ')$$

showing that the design is balanced.

Conversely, let the design be balanced, i.e.,

$$F = \theta\,(I - v^{-1}JJ')$$

on equating the traces of the matrices on both sides, we have (using 2.1).

$$vr - vrq^{-1} - vrp^{-1} + r = \theta\,(v\text{-}1)$$

giving $\theta = vr\,(1 - q^{-1} - p^{-1} + v^{-1})\,/\,(v\text{-}1)$. Then,

$$rI - N_1N_1'/q - N_2N_2'/p + (r/v)\,JJ' = vr(1 - q^{-1} - p^{-1} + v^{-1})(v-1)^{-1}(I - v^{-1}JJ')$$

which gives,

$$pN_1N_1' + qN_2N_2' = [r(p+q) - \mu]\ \mathrm{I} + \mu\, JJ'$$

where, $\mu\,(v\text{-}1) = r\,(2pq - p - q)$

completing the proof of the theorem.

Latin square and Youden square designs are trivial examples of balanced row-column designs. The following design with 9 treatments, 6 rows and 6 columns, reported by Kshirsagar [5] is also balanced.

1	5	6	7	8	9
2	7	4	6	5	3
3	4	2	1	9	8
6	8	9	4	2	7
7	2	1	9	3	5
8	1	5	3	6	4

Table below gives some values of p,q,v and r which satisfy the necessary conditions for a design to be balanced where p and q both are less than v.

v	p	q	r
15	9	10	6
16	4	12	3
21	7	15	5
25	10	10	4
27	9	12	4

However, since these values satisfy only the necessary conditions, namely $vr=pq$ and $\mu(v\text{-}1)=r\,(2pq\text{-}p\text{-}q)$, it is not known whether these designs can be constructed or not. The design reported by Kshirsagar [5] is possibly the only existent balanced row-column design available in the literature for $r \leq 10$ with $p \leq v$ and $q \leq v$.

ACKNOWLEDGEMENT

The help and guidance received from Dr. A. Dey, Senior Professor, IASRI, New Delhi, in pereparation of this paper is gratefully acknowledged.

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Journal of Natural & Physical Sciences Vol. 11 (1997) 47-54

ON ROBUST EXPERIMENTAL DESIGNS WITH RANDOM EFFECT MODEL

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(Received 30.5.1996)

ABSTRACT

Gopalan and Dey [6] have developed a criterion, on the lines similar to one suggested by Box and Draper [3]) for characterising the robust experimental designs assuming a fixed effect model or model I (Eisenhart [5]). In this paper a criterion for characterising robust experimental designs assuming a random effect model or model II (Eisenhart [5]) has been developed and the robustness of the designs eliminating heterogeneity in one direction has been studied.

Mathematics Subject Classification (1991) : 62 K 10

Keywords : Random effect modelm, robustness, idempotent matrices, eigen values, BIB designs, equireplicate, PBIB designs, heterogeneity, dispersion matrix, normal equations, incidence matrix.

INTRODUCTION

Box [2] has enumerated a number of criteria for judging a good response surface design for fitting an empirical interpolation function $Y=X\beta + e$ where $Y=(y_1, y_2, ..., y_n)'$ is an nx1 vector of response observations, X is an nxp matrix of predictor variable observations, β is a px1 vector of parameters to be estimated and e

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is an nxl vector of residuals. One of the criterion is that the design should be insensitive to the wild observations. Box and Draper [3] have derived an appropriate numerical measure of a design's desirability in relation to the insensitivity to wild observations. Box and Draper [3] have derived an appropriate numerical measure of a design's desirability in relation to the insensitivity to wild observations. Box and Draper [3] have shown that in order that the predicted response y at any given point be insensitive to the outlier, the quantity $r = \sum_u r^2_{uu}$ should be minimised, where r_{uu} is the u-th diagonal element of the matrix $R = X(X'X)^{-1} X$. The designs which minimise r are called *robust*.

EXPERIMENTAL DESIGN WITH RANDOM EFFECT MODEL

Let us consider a model with random effect

$$Y = \mu J + \pi\alpha + U\beta + e$$

where $Y = (y_1, \ldots, y_n)'$ is vector of response of the i-th treatment in j-th block, μ is the general mean effects, $\alpha = (\alpha_1, \ldots, \alpha_v)'$ is the vector of treatment effects, $\beta = (\beta_1, \ldots, \beta_b)'$ is the vector of block effects, π' is the treatments vs block incidence matrix, U is the observations vs block incidence matrix and $J = (1, \ldots, 1)'$. Here β is assumed to be the vector of random block effects normally distributed with mean vector O and dispersion matrix $I\,\sigma_b^2$ and e are independent and normally distributed with mean vector O and dispersion matrix $I\,\sigma^2$, I is an identity matrix. The above model may be rewritten as

$$Y = X\theta + e_1$$

where $e_1 = U\beta + e$, $X = [J : \pi_{n,v}]$ and $\theta = (\mu, \alpha)'$

we have,

$$D(e_1) = UU'\,\sigma_b^2 + I\sigma^2$$
$$= Z^{-1}\,\sigma^2 \quad \text{(say)}$$

where Z^{-1} is the variance-covariance matrix and is function of $\Gamma = \sigma^2_b/\sigma^2$ which is assumed to be fixed and known. Application of usual least square technique to estimate the vector of parameters yields the normal equation,

$$\theta = (X'Z^{-1}X)^{-}X'Z^{-1}Y$$

where $(X'Z^{-1}X)^{-}$ is a generalized inverse (g-inverse) of $X'Z^{-1}X$ satisfying $X'Z^{-1}X(X'Z^{-1}X)^{-}X'Z^{-1}X=X'Z^{-1}X$. It is well known that in the absence of any outlier $R^2_{0}/(n-m)$ is an unbiased estimator of σ^2 where,

$$R^2_{0} = Y'[Z^{-1}-Z^{-1}X(X'Z^{-1}X)^{-}X'Z^{-1}]Y$$

m = rank (X) and n is the number of observations.

When u-th observation has added to it a quantity c making it an outlier, the bias or discrepancy in estimating σ^2 through $R^2_{0}{}^{*}$ is $d_u = c^2 a_{uu}/(n-m)$ where R^2_{0*} is the residual sum of squares in the presence of the outlier and a_{uu} is the u-th diagonal element of the matrix

$$\begin{aligned} A &= Z^{-1}-Z^{-1}X(X'Z^{-1}X)^{-}X'Z^{-1} \\ &= Z^{-1}-D(Z^{-1}X'\hat{\theta}) \end{aligned}$$

where D stands for the dispersion matrix.

If it were equally likely that c could occur with any of the n observations, giving rise to $d_1,\ldots,d_n$, the average discrepancy would be

$$\bar{d} = c^2 \sum_{u} a_{uu}/n(n-m)$$

which obviously is not fixed, unlike in the case of the fixed effect model, but is design dependent through A. In order that no unduly large discrepancy in the estimator of σ^2 is caused by the outlier, the average discrepancy should be minimised. A design minimising the average discrepancy, in case of a random effect model, may be called *robust*. We have,

$$\bar{D}(Z^{-1}X'\hat{\theta})/\sigma^2 = Z^{-1}\pi'\,(\pi Z^{-1}\pi')^{-1}\pi Z^{-1}$$

For getting the average discrepancy we need

$$\begin{aligned}\text{trace } Z^{-1}\pi'\,(\pi Z^{-1}\pi')^{-1}\pi Z^{-1} &= \text{trace } (\pi Z^{-1}\pi')^{-1}\pi Z^{-1}Z^{-1}\pi' \\ &= \text{trace } (\pi Z^{-1}\pi')^{-1}\pi Z^{-2}\pi'\end{aligned}$$

We examine below, robustness of some block designs based on the criterion discussed above.

BLOCK DEISGNS : In case of a block design,

$$Z^{-1}=\text{diag}[aI+bJ,\ldots,aI+bJ]$$

and $\pi Z^{-1}\pi' = ar^d+bNN'$

so, $\text{trace } (\pi Z^{-1}\pi')^{-1}\pi\, Z^{-2}\pi' = \text{trace}[ar^d+bNN'][a^2r^d+b\,(2a+kbNN')]$

where r^d is a diagonal matrix with elements of the replication vector r as the diagonal elements and N is the treatment vs block incidence matrix, a and b are constants and k is the block size. Following Bose and Mesner [1], NN' can be decomposed in terms of the eigen values and the corresponding idempotent matrices of the algebra generated by the association matrices. $NN'=\sum_{i=0}^{m}\theta_i L_i$ where θ_i's are the given values of the matrix NN' and L_i's are the corresponding idempotent matrices. Assuming the design equireplicate we have,

$$\text{trace } (\pi Z^{-1}\pi')^{-1}\pi Z^{-2}\pi' = \sum_{i=0}^{m} [a^2r+b(2a+bk)\theta_i]/(ar+b\theta_i)] \qquad (2.1)$$

showing that the average discrepancy is not fixed, as in the case of fixed effect model, but design dependent through θ_i's. In order that the average discrepancy is minimum, the trace given in (2.1) should be maximum subject to the condition that the sum of all the eigen values is constant. So maximising (2.1) subject to the condition

that $\sum_{i=0}^{m} \theta_i = vr$, we get,

$$\theta_i = rb^{-1} + \sqrt{[r(b-c)/\phi]} \qquad (2.2)$$

where $c = b\,(2a+kb)$ and ϕ is the Lagrangian multiplier.

(a) **BIB Designs** : Let us consider a BIB (v,b,r,k,λ) design . Here

$\theta_0 = rk$ and $\sum_{i=0}^{m} \theta_i = r(v-k)$. Using (2.2) we get,

$$\theta_i = r-\lambda$$

i.e. each eigen value of NN' of a BIB design, other than rk, should be $(r-\lambda)$ in order that the design is robust, a property which is always satisfied by a BIB design, showing that a BIB design under random effect model is robust.

(b) ***PBIB Designs with two associate classes :*** Let n_0 n_1 and n_2 be the multiplicities of the roots θ_0, θ_1 and θ_2 respectively of NN'. If the design is connected then θ_0 is a simple root and hence $n_0 = 1$ and other multiplicities satisfy

$$n_1+n_2 = v-1$$

and trace $(NN') = rk+n_1\theta_1+n_2\theta_2$

Maximising (2,1) subject to the condition that sum of eigen values is constant, yields

$$\theta_1 = rv/(v-1) \qquad (2.3)$$

So, a PBIB design assuming a random effect model may be characterised as robust if (2.3) is satisfied. This condition has been throughly checked for all the PBIB designs catalogued by Clatworthy [4] and only two PBIB designs with the following parameters :

(i) $v = 4, b = 12, r = 6, k = 2, \lambda_1 = 4, \lambda_2 = 1$

(ii) $v = 9, b = 24, r = 8, k = 3, \lambda_1 = 1, \lambda_2 = 3$

satisfy the condition (2.3). These designs are R4 and LS13 of Clatworthy [4]. The robustness criterion $\tau = \theta_1 - vr/(v-1)$ for some of the PBIB designs has been tabulated in the Annex. For robust designs the value of τ will be equal to zero

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ANNEX

Table showing the value of the robustness criterion (τ) for some of the PBIB designs

v	r	k	b	λ_1	λ_2	τ	v	r	k	b	λ_1	λ_2	τ
6	2	4	3	2	1	0.4	8	9	6	57	9	1	6.8
6	4	4	6	4	2	0.8	42	10	6	70	10	1	7.8
6	6	4	9	6	3	1.2	10	4	8	5	4	3	2.4
6	8	4	12	8	4	1.6	10	8	8	10	8	6	4.9
6	10	4	15	10	5	2.0	12	2	8	3	2	1	1.8
8	3	4	6	3	1	0.6	12	4	8	6	4	2	3.6
8	6	4	12	6	2	1.2	12	6	8	9	6	3	5.4
8	9	4	18	9	3	1.8	12	8	8	12	8	4	7.3
10	4	4	10	4	1	1.6	12	10	8	15	10	5	9.1
10	8	4	20	8	2	3.1	12	10	8	15	10	6	2.1
12	5	4	15	5	1	2.4	14	4	8	7	4	2	0.3
12	10	4	30	10	2	5.1	14	8	8	14	8	4	0.6
14	6	4	21	6	1	3.5	16	3	8	6	3	1	2.8
16	7	4	28	7	1	4.5	16	6	8	12	6	2	9.6
18	8	4	36	8	1	5.5	16	7	8	14	7	3	0.5
20	9	4	45	9	1	6.5	16	9	8	18	9	3	14.4
22	10	4	55	10	1	7.5	18	8	8	18	8	3	1.5
8	3	6	4	3	2	1.4	20	4	8	10	4	1	7.8
8	6	6	8	6	4	2.9	20	6	8	15	6	2	1.7
8	9	6	12	8	6	4.2	20	8	8	20	8	2	15.6
9	2	6	3	2	1	0.8	24	5	8	15	5	1	10.8
9	4	6	6	4	2	1.5	24	10	8	30	10	2	13.5
9	6	6	9	6	3	2.2	26	4	8	13	4	1	7.8
9	8	6	12	8	4	3.0	26	8	8	26	8	2	3.7
9	10	6	15	10	5	3.7	28	6	8	21	6	1	13.8
10	6	6	10	6	3	0.7	32	5	8	20	5	1	2.2
12	3	6	6	3	1	2.7	32	7	8	28	7	1	16.8
12	5	6	10	5	2	0.5	32	10	8	40	10	2	3.7
12	6	6	12	6	2	5,5	36	8	8	36	8	1	19.8
12	9	6	18	9	3	8.2	40	9	8	45	9	1	22.8
12	10	6	20	10	4	1.1	44	10	8	55	10	1	25.8
14	3	6	7	3	1	0.8	50	8	8	50	8	1	5.9
14	6	6	14	6	2	1.5	56	9	8	63	9	1	7.0
14	9	6	21	9	3	2.3	12	3	9	4	3	2	0.3
15	4	6	10	4	1	4.7	12	6	9	8	6	4	0.6
15	8	6	20	8	2	9.4	12	9	9	12	9	6	0.8
18	4	6	12	4	1	1.8	15	6	9	10	6	3	2.6
18	5	6	15	5	1	6.7	18	5	9	10	5	2	13.7
18	8	6	24	8	2	3.3	18	10	9	20	10	4	7.4
18	10	6	30	10	2	13.4	21	3	9	7	3	1	2.8
20	9	6	30	9	2	4.5	21	6	9	14	6	2	5.7
21	6	6	21	6	1	8.7	21	9	9	21	9	3	8.5
24	7	6	28	7	1	0.2	27	4	9	12	4	1	4.8
26	6	6	26	6	1	3.8	27	8	9	24	8	2	9.7
27	8	6	36	8	1	12.7	30	9	9	30	9	2	11.7
30	7	6	35	7	1	4.8	39	6	9	26	6	1	8.8
30	9	6	45	9	1	14.7	45	7	9	35	7	1	10.8
33	10	6	55	10	1	16.7	57	9	9	57	9	1	14.8

v	r	k	b	λ_1	λ_2	τ	v	r	k	b	λ_1	λ_2	τ
63	10	9	70	10	1	16.8	6	4	3	8	0	2	4.8
12	5	10	6	5	4	3.5	6	5	3	12	0	3	7.2
12	10	10	12	10	8	6.9	6	8	3	16	0	4	9.6
15	2	10	3	2	1	2.8	6	10	3	20	0	5	12.0
15	4	10	6	4	2	5.7	9	3	3	9	0	1	3.4
15	6	10	9	6	3	8.6	9	6	3	18	0	2	6.7
15	8	10	12	8	4	11.4	9	9	3	27	0	3	10.1
15	10	10	15	10	5	14.3	12	4	3	16	0	1	4.4
18	10	10	18	10	5	0.6	12	8	3	32	0	2	8.7
20	3	10	6	3	1	6.8	15	5	3	25	0	1	5.4
20	6	10	12	6	2	13.7	10	4	5	8	0	2	4.4
20	9	10	18	9	3	20.5	10	6	5	12	0	3	6.7
20	9	10	18	9	4	0.5	10	8	5	16	0	4	8.9
22	5	10	11	5	2	0.8	10	10	5	20	0	5	11.1
22	10	10	22	10	4	1.5	15	6	5	18	0	2	6.4
25	4	10	10	4	1	10.8	15	9	5	27	0	3	9.6
25	8	10	20	8	2	21.7	20	4	5	16	0	1	4.2
30	5	10	15	5	1	14.8	20	8	5	32	0	2	8.4
30	10	10	30	10	2	29.7	25	5	5	25	0	1	5.2
35	6	10	21	6	1	18.8	14	6	7	12	0	3	6.4
40	7	10	28	7	1	22.2	14	8	7	16	0	4	8.6
42	5	10	21	5	1	2.9	14	10	7	20	0	5	10.8
42	10	10	42	10	2	5.8	21	6	7	18	0	2	6.3
45	8	10	36	8	1	26.8	21	9	7	27	0	3	9.5
50	6	10	30	6	1	3.9	28	8	7	32	0	2	8.3
50	9	10	45	9	1	30.8	9	7	7	9	6	5	3.9
55	10	10	55	10	1	34.8	12	7	7	12	6	2	17.4
82	10	10	82	10	1	7.9	12	7	7	12	2	4	6.6
4	2	2	4	0	1	2.8	12	7	7	12	3	4	6.7
4	4	2	8	0	2	5.3	14	7	7	14	6	3	0.5
4	6	2	12	0	3	8.0	18	7	7	18	6	1	23.6
4	8	2	16	0	4	10.7	10	6	2	30	1	0	4.7
4	10	2	20	0	5	13.3	10	3	2	15	0	1	7.3
6	3	2	9	0	1	3.6	10	3	3	10	1	0	5.7
6	6	2	18	0	2	7.2	10	6	3	20	2	0	11.3
6	9	2	27	0	3	10.8	15	4	3	20	1	0	7.7
8	4	2	16	0	1	4.6	15	3	3	15	0	1	9.2
8	8	2	32	0	2	9.1	10	3	5	6	1	2	8.3
10	5	2	25	0	1	5.6	10	6	5	12	3	2	16.7
10	10	2	50	0	2	11.1	10	6	5	12	2	4	22.6
12	6	2	36	0	1	6.6	15	2	5	6	1	0	7.8
14	7	2	49	0	1	7.5	15	4	5	12	2	0	15.7
16	8	2	64	0	1	8.5	10	7	7	10	5	4	1.2
20	10	2	100	0	1	10.5	28	2	7	8	1	0	11.9
6	2	3	4	0	1	2.4							

Journal of Natural & Physical Sciences Vol. 11 (1997) 55-64

A BIVISCOSITY MODEL OF CONVECTIVE STABILITY FOR BLOOD FLOW BETWEEN PARALLEL PLANTES

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(Received 30.10.1991, Revised 12.11.1996)

ABSTRACT

An analysis has been carried out for establishing the stability criterion for a vertical layer of blood treated as a non-Newtonian fluid (biviscosity fluid) heated from below filled in a tube formed by infinite parallel plannes. For equilibrium, a constant vertical temperature gradient is maintained in the tube. The boundaries of the plastic core region, moving with constant velocity, are determined for different values of Rayleigh number. Stability boundaries are investigated. For stable equilibrium, critical value of Rayleigh number has been analysed. Unstable equilibrium exists only under finite perturbation velocities, whose value are different for different values of parameters x.

Keywords : Biviscosity, Rayleigh number, Convective stability

INTRODUCTION

Blood is a complex fluid. Little is known definitely about the effect of the temperature on human blood. From a physiological standpoint the ideal temperature for blood is 37^0C. However, blood can function with reduced ability to absorb oxygen and deabsorb carbon-dioxide in a range of temperatures from 30^0C to 45^0C. Blood can survive for a long period of time if frozen (-5^0C to -10^0C) and

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thawed in an appropriate manner. Above 45^{0}C, the proteins in the blood will start to precipitate out of the plasma and coagulate. Enzymes in the blood do not function well above 45^{0}C. Change of blood temperature of this degree is unlikely to occur within a body as the normal body temperature regulation system will active in a attempt to reverse this effect. Many situations now exist in which the blood is removed from the body for processing (for example, oxygenation, hemodialysis etc) and again returned to the body. While in the body, temperature control of blood becomes important in order to prevent damage. The information about the temperature field for blood flowing in a tube is therefore of practical significance.

It has been shown experimentally by Charm and Kurland [4] that the rheological behaviour of blood can be approximated by the Casson's relation strongly suggesting the non-Newtonian nature of blood flow, particularly at low rates of shear. The flow of blood in a circular tube assuming it to be a Casson fluid has been investigated by Jones [5]. Flow of Casson fluid in non circular ducts has been investigated by Batra and Koshy [3]. Natural convective flow characteristics for a Casson fluid in a vertical uniform heat flux duct have been investigated by Batra and Mohan [2]. Effect of non-uniform inlet velocity profile on the heat transfer in the entrance region of a Casson fluid between parallel plates has been considered by Batra and Koshy [1].

In this paper, an attempt has been made to establish the criteria for the stability analysis of a vertical layer of blood treated as a biviscosity fluid heated from below filled in a tube formed by infinite parallel planes kept at a distant 2d apart. In the tube a constant vertical temperature is maintained. The problem is investigated by considering the solution in the three regions consisting of two viscous flow regions separated by a plastic region moving with constant velocity.

FORMULATION OF THE PROBLEM

A layer of a non-Newtonian fluid filled in a tube formed by infinite parallel planes located at a distance 2d apart is heated from below. The origin is chosen at the middle of the tube and cartesian co-ordinate system (x, y) is used. The constitutive equation for blood considered as a biviscosity model is given by Nakamura, *M*. and *T*. Swada (1988) as

$$\left.\begin{aligned} \tau_{\mathrm{ij}} &= 2(\mu_\beta + \frac{p_y}{\sqrt{2\pi}}\, e_{ij}) \qquad && \pi \geq \pi_c, \\ &= 2(\mu_\beta + \frac{p_y}{\sqrt{2\pi_c}}\, e_{ij}) \qquad && \pi < \pi_c, \end{aligned}\right\} \tag{1}$$

where $\beta = \mu_\beta \sqrt{2\pi_c}/p_y$. (2)

Equations governing the steady laminar incompressible flow for a biviscous fluid are given by

$$\rho \left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right] = \rho X + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}, \tag{3}$$

$$\rho \left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \right] = \rho Y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}, \tag{4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{5}$$

$$\rho\, C_p [\, u\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y}\,] = k\, [\, \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\,] + \mu\, \Phi, \tag{6}$$

where u, v are velocity components and X, Y the body forces, c_p is the specific heat at constant pressure and K is thermal conductivity, Φ is dissipation function given by

$$\Phi = \tau_{ij}\, e_{ij} = [\frac{\partial u}{\partial x}]^2 + [\frac{\partial v}{\partial y}]^2 + \frac{1}{2}\, [\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\,]^2 \tag{7}$$

and

$$\tau_{xx} = 2\, [\mu_\beta + \frac{p_y}{g}]\, \frac{\partial u}{\partial x}, \quad \tau_{xy} = [\mu_\beta + \frac{p_y}{g}]\, [\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}],$$

$$\tau_{xy} = 2\, [\mu_\beta + \frac{p_y}{g}]\, \frac{\partial v}{\partial y}, \tag{8}$$

when

$$g = \sqrt{2\pi}, \quad \pi \geq, \pi_C$$
$$= \sqrt{2\pi_C}, \quad \pi <, \pi_C$$

The problem is investigated under the following assumptions :

(1) A constant vertical temperature gradient is maintained.

(2) In equilibrium, there is steady laminar incompressible flow of the fluid.

(3) Heat produced by viscous dissipation and conduction is negligible.

(4) For relatively low rates of flow, pressure is a function of y alone.

(5) Horizontal velocity component and vertical velocity gradient are negligible.

Under the above assumptions equations (3) and (5) are identically satisfied and equations (1), (4) and (6) reduce to

$$\tau = [\mu_\beta + \frac{p_y}{\sqrt{2\pi}}]\,\frac{dv}{dx}\,, \tag{9}$$

$$\frac{1}{p}\;\frac{dp}{dy} = -g + \frac{1}{\rho}\;\frac{\partial\,\tau_{xy}}{\partial\,x} \tag{10}$$

$$k\,\frac{\partial^2 T}{\partial\,x^2} = -\,\rho\,D\,c_p v \tag{11}$$

The equations are to be solved under the following boundary conditions :

$$v(\pm d) = 0. \tag{12}$$

$$T(\pm d) = 0. \tag{13}$$

also the equation must satisfy the close flow condition given by

$$\int_{-d}^{d} v\,d\,x = 0. \tag{14}$$

SOLUTION OF THE PROBLEM

If ρm is the density at mean fluid temperature T_m, then we have

$$k\,\frac{dp}{dy} = -\,\rho_m\,g\,. \tag{15}$$

Equation of state for small temperature difference is

$$\rho_m = \rho\,[1-\gamma(T_m - T)]\,. \tag{16}$$

where γ is the coefficient of thermal expansion. Substituting (15) in equations (10) and equation (16) in equation (8), we get

$$\frac{1}{\rho}\frac{d\tau}{dx} + g\,\gamma\,\theta = 0 \quad , \tag{17}$$

$$\alpha_1\,\theta'' = -\,DV \quad , \tag{18}$$

where $\theta = T - T_{m},\quad \alpha_1 = \dfrac{k}{\rho c_p}$ is thermal diffusivity. Non-dimensionalising equation (9), (17) and (18), we get

$$\tau' = (1+\beta^{-1}) + \frac{dv'}{dx}, \tag{19}$$

$$\frac{d\tau'}{dx'} + R\,\theta = 0 \tag{20}$$

$$\frac{d^2\theta'}{dx'^2} = -v' \tag{21}$$

where

$$x' = \frac{x}{d},\ v' = \frac{vd}{\alpha_1},\ \theta' = \frac{\theta}{Dd},\ = \frac{\tau d^2}{\mu_\beta \alpha_1}\ ,$$

$$R = \frac{\rho g\ r\,d^4 D}{\alpha_1 \mu_\beta}.$$

where (12), (13) and (14) in dimensionless form become

$$v'(\pm 1) = 0 \quad , \tag{22}$$

$$\theta'(\pm 1) = 0 \quad , \tag{23}$$

$$\int_{-1}^{1} v'\,dx' = 0. \tag{24}$$

Eliminating τ' from equation (19) and (20) we get

$$\frac{d}{dx}\left[(1+\beta^{-1}) + \frac{dv'}{dx}\right] + R\,\theta' = 0,$$

$$(1+\beta^{-1})\frac{d^2v'}{dx'^2} = -\,R\,\theta' \quad . \tag{25}$$

Again eliminating v' from equations (21) and (25) we get

$$[(1+\beta^{-1}) \quad \frac{d^4\theta'}{dx'^4} = R\,\theta' \ . \tag{26}$$

Solving the differential equation (26), we get

$$\theta' = c_1\, e^{\alpha x'} + c_2 e^{-\alpha x'} + c_3 cos\ \alpha\, x' + c_4 \sin \alpha\, x' \ , \tag{27}$$

where, $\alpha = \dfrac{R}{1+\beta^{-1}}$,

Substitute the value of $\theta\,'$ in (21) we get v'

$$v' = \frac{d^2\theta'}{dx'^2} = -(\alpha^2 e^{\alpha x'} + \alpha^2 e^{-\alpha x'} - c_3\, \alpha^2 \cos \alpha x' - c_4\, \alpha^2 \sin \alpha x') \tag{28}$$

where c_1, c_2, c_3 and c_4 are constants.

Assuming $c_4 = \lambda$ (constant), for a particular model we have solved the simultaneous equations (27) and (28) under the boundary conditions (22) to (24) for many solutions of v' and θ' for different values of λ. The constants c_1, c_2 and c_3 have been obtained as

$$c_1 = -\frac{\lambda \sin \alpha\, \{(e^{\alpha} - e^{-\alpha})\,(\sin \alpha + \cos \alpha) + 2 \sin \alpha\, e^{-\alpha}}{\cos \lambda\, (e^{\lambda} - e^{-\lambda})^2 + \sin \lambda\, (e^{2\lambda} - e^{2\lambda})} \ , \tag{29}$$

$$c_2 = \lambda \sin \alpha\, [\frac{1}{(e^{\alpha} - e^{''})} - \frac{\{(e - e^{\alpha})^{-\alpha}\,(\sin \alpha + \cos \alpha) + 2 \sin \alpha\, e^{-2\alpha}}{\cos \alpha\ (e^{\alpha} - e^{-\alpha})^2 + \sin \alpha\, (e^{\alpha}(e^{\alpha} - e^{-\alpha}) + 1 - e^{-2\alpha})}], \tag{30}$$

$$c_3 = \frac{(C_1 + C_2)\,(e^{\alpha} - e^{-\alpha})}{2 \sin \alpha}, \tag{31}$$

RESULT AND DISCUSSION

On the basis of foregoing analysis and numerical calculations the following conclusion can be drawn for different models of blood. Velocity and temperature profiles have been shown for different values of Rayleigh number and biviscous parameter λ, which is responsible for change in nature of velocity and temperature profiles as compared with power law model Katiyar et. al [6].

1. From figs (1) and (2) is is evident that the velocity increases for increasing values of R for different values of x, the width of planes. the velocity changes significantly for increasing values of the biviscous parameter, which shows the better efficiency of the model as compared with other models as was expected.

2. Figs. (3) and (4) show that the temperature changes initially with x, but attains uniformity for all width for value of biviscous parameter. There is a uniform delay in temperature regulation system when biviscous parameter increases which is physiologically true for biothermal phenomenon in circulatory system.

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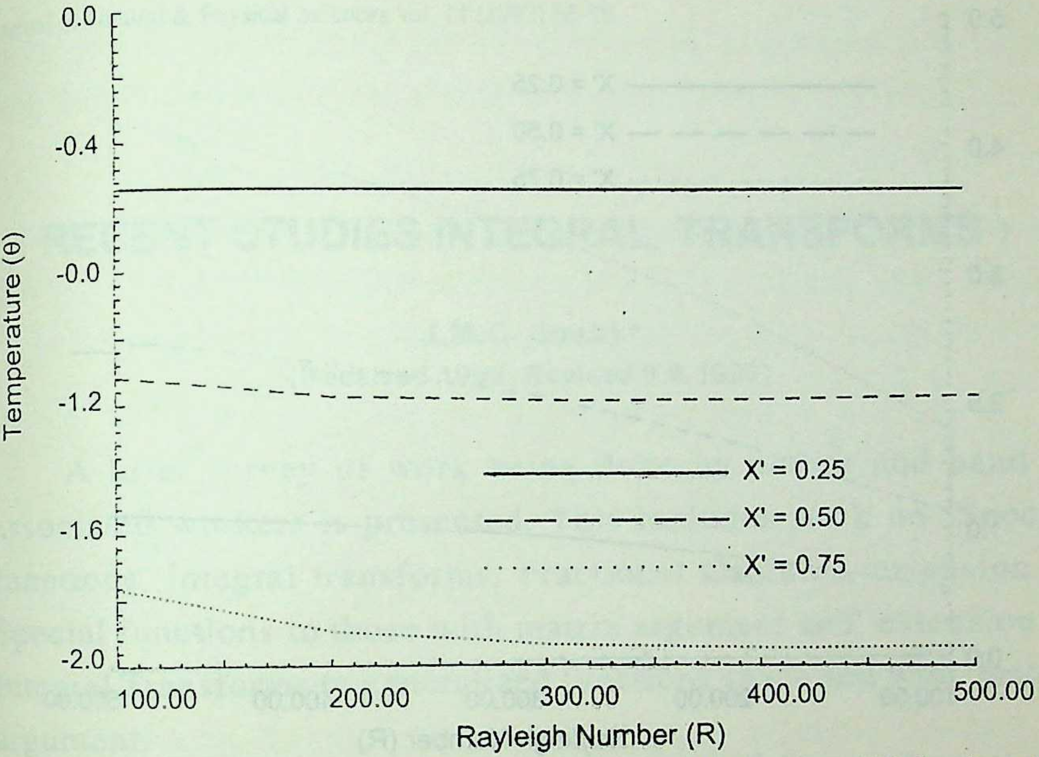

Fig. 1 Variation of Temperature with Rayleigh Number (R) for λ = 1.0

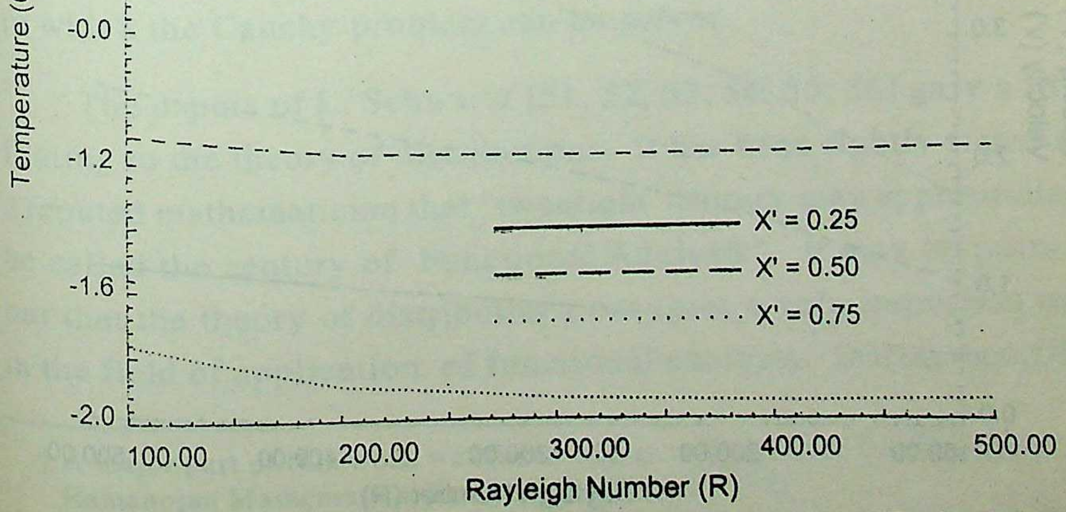

Fig. 2 Variation of Temperature with Rayleigh Number (R) for λ = 3.0

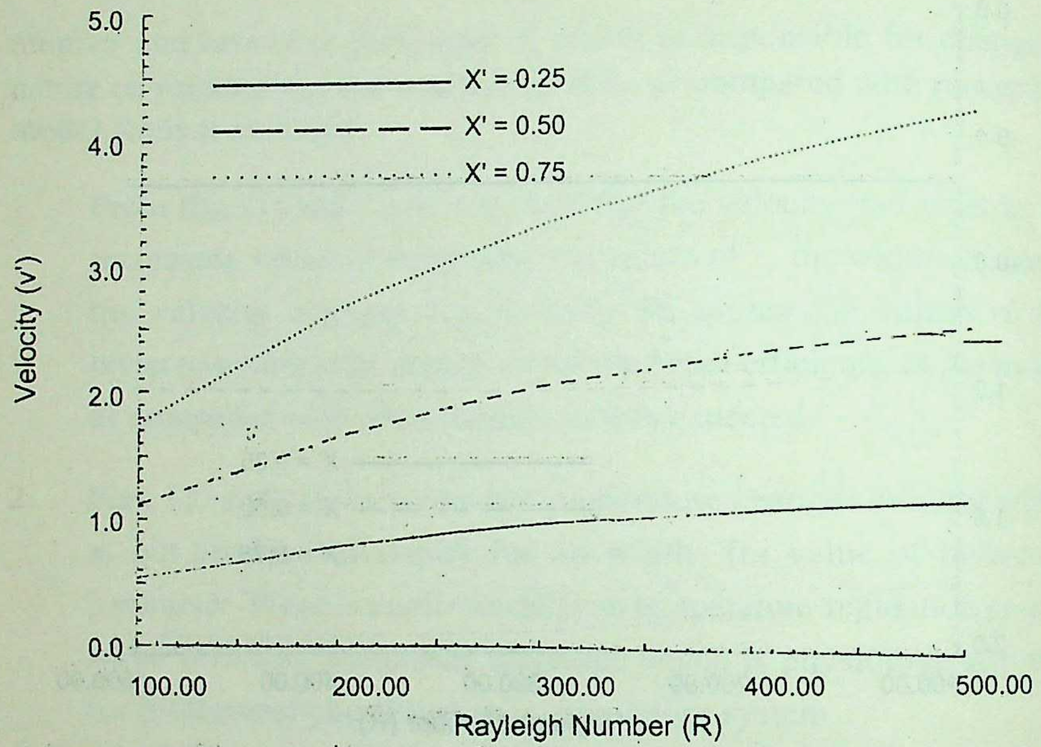

Fig. 3 Variation of Velocity with Rayleigh Number (R) for $\lambda = 1.0$

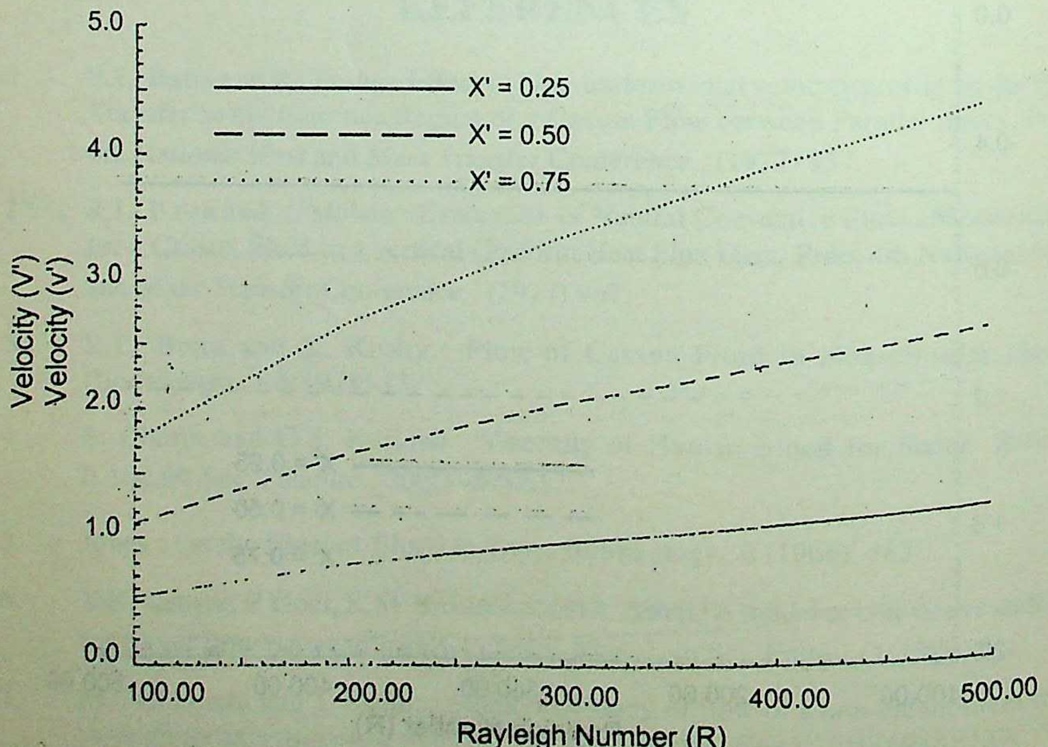

Fig. 4 Variation of Velocity with Rayleigh Number (R) for $\lambda = 3.0$

Journal of Natural & Physical Sciences Vol. 11 (1997) 65-78

RECENT STUDIES INTEGRAL TRANSFORMS †

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(Received 1996, Revised 8.8.1997)

A brief survey of work being done by author and band of associated workers is presented. This includes work on Special functions, Integral transforms, Fractional Calculus, extension of Special functions to those with matrix argument and extension of Integral Transforms to generalized functions space and with matrix argument.

The theory of integral transforms in the classical sense can be traced back to about 200 years. But the theory of distributions is only seventy five years old. Some elements of the theory may be found in a paper of Sobolev [50]. He actually tried to generalize the function concept and the classical operations to a larger field in which the Cauchy problem can be solved.

The papers of L. Schwartz [51, 52, 53, 54, 55, 56] gave a firm footing to the theory of distributions. It has been rightly stated by a reputed mathematician that "twentieth century may appropriately be called the century of Functional Analysis". It may be pointed out that the theory of distributions occupies a very important role in the field of application of functional analysis. During last fifty years the phenomenal growth in the theory of partial differential equations is the result of theory of distributions which has acted as a catalytic agent.

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† A major part of this work was presented at tenth Annual Conference of the Ramanujan Mathematical Society (Rishikesh 1995).

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Besides Schwartz others such as Siroski [57] and Minkusinski, Temple [58], Ravetz (59). Bremermann [60] Kothe [61], Tillmann [62] and Sato [63] gave other approaches to distribution theory.

It all started with giving mathematically accepted definition to Dirac's Delta function defined by

$\delta(x) = 0$ for $x \neq 0$

and $\int_0^\infty \delta(x)dx = 1$

This function was used to represent mathematically the concept of a shock, like the impact of hammer in mechanics or a large voltage of very small duration in electrical engineering. But it is not a function in the strict sense. For in the theory of integrals if a function is zero almost everywhere in an interval, its integral over that interval is also zero. Actually Delta function is a functional.

As an elementary definition we denote by K the set of all real functions $\phi(x)$ with continuous derivatives of all orders and with bounded support (The support of a continuous function $\phi(x)$ is the closure of the set on which $\phi(x) \neq 0$)

We call these functions the test functions and call K the space of test functions. We then define a generalized function as any linear continuous functional defined on K. The terms 'distributions' and generalized functions are used by most of the mathematicians as synonymous but $A.H.$ Zomanian makes a distinction between these two terms. The term 'distribution' is reserved for members of a subspace of a space of generalized functions f whose elements can be identified with f in a one-one fashion. Thus every distribution is a generalized function but not conversely [64] .

An interesting property of distributions is that each distribution is differentiable & derivative, in general, is not a function but a distribution. Also where as in classical analysis continuous functions are not necessarily differentiable but in space of distributions each continuous function becomes differentiable. Its derivative is a distribution. For instance the non differentiable Wierstrass function is differentiable in the sense of distributions but its derivative is not a function but a distribution. Every primitive of a distribution is a distribution.

There are mainly three ways in which a classical integral transform, say,

$$f(s) = \int_I K(s,t)\, \phi(t)\, dt$$

can be generalized.

In the first instance kernel of the trnasform can be replaced by a more general Special function. For example the kernel e^{-st} of well-known Laplace transform has been replaced by $(st)_1 F_1(\beta+\eta+1;\alpha+\beta+\eta+1;-st)$ as has been done by me in introducing Dr. S.M. Joshi generalized Laplace transform (named after my revered father Dr. S.M. Joshi, who inspired me for research) [4,5,7,8,26,64]. Though kernel of the above transform was obtained by Erdelyi by the use of operators of fractional integration, as a special case of a more general kernel, but as a transform it was introduced by me by giving its existence and convergence theorems. Abelian theorems, complex inversion theorems and real inversion theorems were developed by me for the same. [1,2,3].

Similarly the widely known Stieltjes transform

$$f(s) = \int_0^{\infty} \frac{dt}{s+t} f(t)dt$$

was generalized by me with the nomenclature Dr. S.M. Joshi generalized Stieltjes transform [1,4,6,7] in the form

$$S^{ac}\, b(f) = \frac{\Gamma(a)\Gamma(c)}{\Gamma(b)\, s} \int_0^{\infty} (t/s)^{c-1}\ \ 2^{F}1\ \ (a,c;b;-\frac{t}{s})\, f(t)\, dt.$$

In this notation Dr. S.M. Joshi generalized Laplace transform can be written $S_b^a(f)$

Dr. S.M. Joshi generalized Stieltjes transform is well established in literature [6, 65, 66 (p. 132)]. Similarly Dr. S.M. Joshi Generalized Hankel transform

$$Sf = \int_0^{\infty} e^{ist}\ {}_1F_1\ (a;\ b;\ -\ 2ist)\, f(t)dt$$

was introduced by us in 1989 [33].

Also Beta transform was introduced in [68] by us.

It may be pointed out that Dr. S.M. Joshi generalized transform was first discussed by the author in [1958] and its convergence theorems [7], Abelian the theorems [4,7], Real inversion theorems [2], Complex inversion theorem [3] were developed besides introducing Dr. S.M. Joshi generalized Laplace transform of two variables [4,7]. Dr. S.M. Joshi generalized transform as a convolution transform, Fractional integration and Dr. S.M. Joshi generalized Laplace transform [5] and Fractional integration and generalized Hankel transform [1, 67] were discussed by me.

Dr. S.M. Joshi generalized transform was extended to distributions in a slightly changed form in [26] .

It may be pointed out that there are mainly three ways in which a classical integral transform may be extended to generalized functions. In the first method the kernel containing testing function space say S(I), is constructed, imposing as many restrictions on it as necessary. In the second way kernel is to be a function of real variable, say x istead of s. We then construct a testing function space $S(I)$ not necessarily containing the kernel, which is such that $\phi(x) \longrightarrow \overline{\phi(x)}$ is an isomerphism from $S(I)$ to another testing function space $\overline{S(I)}$, when there exists a Parsewal relation of the type

$$\int_I [K(\psi)]\ (t)\ \phi\ (t)dt = \int_I \psi(x)\ \overline{\phi\ (x)}dx$$

with suitable restrictions on ϕ and ψ Then $\overline{S}'(\mathrm{I})$ is the required space of distributions and transform of generalized functions in defined by

$$< Kf, \phi > = < f, \phi >$$

There is yet another method of generalizing a transform with certain kernel K_2 by the help of another transform with kernol K_1 which can be generalized by first method. A form of generalized Stieltjes transform introduced by me was discussed and extended to distributions in [11] and in [14] .

In [33] Dr. S.M. Joshi generalized Hankel transform was extended to distributions. In the same work explicit error terms for asymptotic expansions of Dr. S.M. Joshi generalized Stieltjes transform and Dr. S.M. Joshi generalized Laplace transform were discussed. The researcher also found Abelian theorems, Complex inversion theorems etc. for Dr. S.M. Joshi generalized Hankel transform. Haq, I. discussed the transform

$$f(s) = \int_0^\infty e^{-st} (st)^{C+1} \psi (a-b; 2+c-b; st) \phi(t) dt$$

in generalized function space [23, 32] Nayal [45-48, 67] discussed Laplace-Hardy transform in detail. Upreti, *R*. discussed a moment problems, distributional Beta transform, Gamma type operators [17,28, 68].

As early as 1952 Bochner introduced Bessel function of matrix argument in connection with a lattice point problem [69]. Herz [70] extended his work and generalized the classical special functions of hypergeemetric type to functions of matrix agruments. He gave the integral representation of the hypergeometric function with the help of generalized Laplace and its inverse. One of the more exciting results in the multivariate analysis has been the discovery of zonal polynonials by James [71, 72] Constantine [73] showed that zonal polynonials form a basis of a meaningful generalizations of whole family of hypergeometric functions to matrix argument. Besides others Mathai & Saksena [76] have studied the subject in detail R.M. Joshi discussed [27] the theory of special functions and integral transforms with matrix arguments. In particular we discussed inversion theorem for Laplace transform and extension of Confluent Hypergeometric function of Second kind [31]. We also extended Dr. S.M. Joshi Generalized transform to matrix argument [74] Some other integral representations have also been obtained. Operational calculus for Laplace transform of matrix agrument has been developed by us. Dhami [9] has also studied matrix argument analysis in his thesis. Some theorems for special functions have been discussed by him [15, 16].

Hahn [75] defined basic analogues of well known Laplace transform in 1949. When $q \rightarrow 1$ these reduce to classical Laplace transform Let q be a parametre which, in general, is restricted to the condition $|q| < 1$. Basic analogue or q-analogue of a number 'a' is defined by

$$[a] = \frac{1-q^a}{1-q}$$

When $q \rightarrow 1$, $[a] \rightarrow a$

Similarly q-difference operator and q-integral operators are defined. For definitions etc. see [18]. We have discussed extension of distributional q-analogue of fractional integration operators. Similarly Distributional hasic Laplace and Stiltjes transforms have been developed.

More recently Bisht [77] has studied theory of numerical inversion of integral transforms (including Dr. S.M. Joshi Generalized transforms) discussed above in order to make the same computer oriented as demanded by modern technology. A chapter devoted to evaluating real zeros of the Confluent Hypergeometric function by developing an algorithm and a program in C language. The program runs well under the DOS and Xenix (Unix) environment.

We have tried to study statistically the style problem in literature [13, 21]. Also Dr. S.M. Joshi generalized Laplace transform has been used in Statistics [34, 36, 38]. It has been extended to Lerentz spaces [22, 35]. Q-analogues of P.J. operators (defined in memory of my Respected Mother Smt. Parwavi Joshi) have been developed and discussed in [37].

Other activities under the guidance of the author include research in fixed point theorems [42, 44] by R.P. Pant and on extreme Forms by C.S. Bisht [43]. An important paper (presented at International Conference in Tokyo) [39, 40] indicated the use of fractional calculus to develop a new formula (called S.M.P.J. formula in memory of my Respected Father and Respected Mother) for sum of powers of natural numbers. This has been recently appriciated by Prof. S.C. Woon of Deptt. of Applied Maths & Physics, Cambridge University in his letter to me.

A bibliography of work being done at this centre also appears in [41] which was presented at International conference at Koriyama (Japan).

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Journal of Natural & Physical Sciences Vol. 11 (1997) 79-86

AN ANALYSIS OF WATER BEING SUPPLIED TO GURUKULA KANGRI CAMPUS, HARDWAR

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(Received 29.5.1997; Revised 29.9.1997)

ABSTRACT

The water being supplied from different bodies to the various residential colonies, offices, laboratories etc. situated inside the campus of Gurukula Kangri University, Hardwar, has been analysed for a number of Physico-Chemical and Biological parameters and the results have been interpretated and compared with the standards prescribed by USPHS, IS, WHO and ICMR.

Keywords: Water analysis, Gurukula Kangri Hardwar Campus.

INTRODUCTION

Gurukula Kangri University is situated on Hardwar Delhi highway, just about 6 Km away from the famous Har-ki-pouri in Hardwar.

Keeping in view the water pollution problem, the analysis of water being supplied to the houses, offices, laboratories and cattle houses situated inside the campus of Gurukula Kangri University, was under taken by us with an aim to provide suggestions for the improvement of water quality. There are four main sources of water supplies in the Gurukula Kangri University campus:-

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1. Municipal water supplies to professors' quarters at "Satya Shikshak Sadan" .(constituting the sample being referred to as sample no. 1 in the following pages.)

2. Water being supplied from the tube well adjacent to the vice-Chancellor's office) (being referred to as sample number 2 in the following pages). This water is supplied to the laboratories, departments and office in faculty of science and Registrar's office.

3. Supplies from the tube well situated in Gurukula (i.e. Children wing of university) to " Chhota parivar colony", "Bada parivar colony" and the children's hostel (being referred to as sample No.3 here after).

4. Supplies from the tube well/tank in" Gaushala". This water is being supplied to Gaushala and few houses in the " Chhota parivar colony" (being referred to as sample number (4) here after).

The water is being utilized by campus residents for domestic purposes, gardening, laboratory work, and for their cattle. However besides it, there is one more source of water supplies in university campus and that is from the Ganga canal, which is being utilized for the irrigation purposes in the agricultural fields of the university. This has not been included in the present studies.

MATERIAL AND METHODS

All chemicals used were of A.R./G.R. grade, "A" certificate pipettes were used. All other glassware used were of borosil glass. Composite and integrated samples were collected from different sources. Polyethylene bottles were used for this purpose . However, glass bottles were also used for the collection of samples for the purpose of dissolved oxygen analysis. The samples were analysed for different physico-chemical parameters as per standard procedure[1-3]. Dissolved oxygen was analysed by winkler's method [6].

"Systronics -324" pH meter, "Systronics conductometer-305", "Systronics nephelo-turbidimeter-131" and "Systronics Flame photometer Mediflame-127" were used for measurement of pH, conductance, turbidity, sodium and potassium contents in the samples respectively.

RESULTS AND DISCUSSION

The analysed parameters for the four samples under consideration, are given in table -1. The domestic water supplies as suggested by USPHS and IS [1,4] are shown in the table -2. The IS values, available only for the few parameters, are much higher than those for USPHS, obviously for no good reasons. So some ICMR and WHO [5] values are also shown in table-2.

A perusal of the data and its comparison with the standard values reveals that the colour, odour, taste and turbidity are within the permissible limits. However, the specific conductance reveals that in the sample no. 4 and 3 the concentration of ions is much higher than permitted but it causes no worry because this water is not being used for agricultural purposes. Total hardness is also within permissible limit according to the WHO standards. The results of DO, COD, ammonia, sulphate, nitrite, nitrate, calcium, inorganic phosphorus, chlorides, total solids, total dissolved solids, chromium and mercury are well within the permissible concentrations.

In all samples the magnesium and suspended solids are in excess to the standard value. In sample no. 1 and 4, iron is in excess to the standard value. However sample No.2 and 3 show the iron contents within permissible limit. However, the suspended solids are not in so much high concentration that the sample can be rejected. Higher values for iron and magnesium concentrations are perhaps due to the fact that the water tanks and the distribution pipe lines are very old besides being rarely cleaned and are also reported to be in damaged conditions at a few points.

Table-1
Results of Sample Analysis

S.No.	Parameters	Sample No.1	Sample No.2	Sample No.3	Sample No.4
1.	Temperature	25°C	26°C	25°C	27°C
2.	Color	Colour less	Colour less	Colour less	Colour less
3.	Odour	n.d.	n.d	n.d.	n.d.
4.	Turbidity (NTU)	0.5	2.0	1.0	1.5
5.	pH	7.6	7.3	8.0	7.6
6.	Conductivity (μmhos)	$1.3x10^2$	$0.9x10^2$	$3.2x10^2$	$4.2x10^2$
7.	Dissolved oxygen (mg/L)	6.08	6.40	3.60	4.40
8.	Biochemical oxygen demand(mg/L)	2.84	1.13	1.24	0.35
9.	Chemical oxygen demand (mg/L)	11.2	7.2	9.6	8.0
10.	Hardness (mg/L)	98.0	104	102	128
11.	Calcium (mg/L)	94	82	96	88
12.	Magnesium (mg/L)	40.70	49.95	42.50	52.62
13.	Total dissolved solids(mg/L)	290	240	320	200
14.	Suspended solids (mg/L)	10	10	30	30
15.	Total solids (mg/L)	300	250	350	230
16.	Chloride (mg/L)	24.14	21.30	17.04	17.04

Table-1 Continued

S.No.	Parameters	Sample No.1	Sample No.2	Sample No.3	Sample No.4
17.	Residual Chloride (mg/L)x 10^2	1.775	3.550	1.775	5.325
18.	Oil and Grease (mg/L)	0.0	0.0	0.0	0.0
19.	Sulphate (mg/L)	15.5	16.0	16.5	8.0
20.	Nitrate (mg/L)	0.38	0.27	0.81	0.28
21.	Nitrite(mg/L)	0.051	0.019	0.023	0.051
22.	Inorganic Phosphorus (mg/L)	0.0	0.15	0.11	0.0
23.	Total Phosphorus (mg/L)	0.30	0.32	0.27	0.24
24.	Organic phosphorus(mg/L)	0.30	0.17	0.16	0.24
25.	Ammonia (mg/L)	0.28	0.0	0.0	0.0
26	Total Kjeldahl nitrogen(mg/L)	0.0	0.0	0.0	0.0
27.	Organic nitrogen (mg/L)	0.28	0.0	0.0	0.0
28.	Total alkalinity (mg/L)	235	250	260	245
29.	Acidity (mg/L)	62.5	107.5	72.5	45.0
30	Sodium (mg/L)	53	65	68	41
31.	Potassium (mg/L)	1.3	1.7	1.6	1.6
32.	Chromium (mg/L)	0.050	0.016	0.024	0.040
33.	Iron (mg/L)	0.42	0.23	0.23	0.34
34.	Mercury (mg/L)	0.0	0.0	0.0	0.0
35.	Colliform cells per 100 ml	0.0	6.63	4.53	9.25

n.d. = nothing disagreeable

Table-2
Standards of Water Quality [4,8,9]
(Water For Drinking Purpose)

S.No.	Parameters	USPHS	Bureau of Indian Standards (IS-10500 : 1991)		ICMR	
			Requirement Desirable Limit	Permissible Limit in the absence of alternate source	Highest desirable level	Maximum Permissible level
	Colour	Colourless	5	--	25.0 units	25.0 units
	Odour	Odourless	Unobjectionable	--	Unobjectionable	Unobjectionable
	Taste	Tasteless	Agreeable	10.0	Unobjectionable	Unobjectionable
4.	Turbidity	5.0 NTU	10.0 NTU	10.0 NTU	5.0 JTU	25.0JTU
5.	pH	6.0-8.5	6.5-8.5	No relaxtion	7.0-8.5	6.5-9.2
6.	Dissolved oxygen (mg/L)	4.0-6.0	--	--	--	--
7.	Specific Conductance	300 μmho cm^{-1}	--	--	--	--
8.	Chemical oxygen demand (mg/L)	4.0	--	--	--	--
9.	*Total hardness (mg/L)	--	300.0	600.0	300.0	600.0
10.	Total dissolved solid (mg/L)	500.0	500.0	2000.0	500.0	1500.0**
11.	Suspended solid (mg/L)	5.0	--	--	--	--
12.	Total solid (mg/L)	505.0	--	--	--	--
13.	Chloride (mg/L)	250.0	250.0	1000.0	200.0	1000.0

Table-2 Continued

S.No.	Parameters	USPHS	Bureau of Indian Standards (IS-10500 : 1991) Requirement Desirable Limit	Bureau of Indian Standards (IS-10500 : 1991) Permissible Limit in the absence of alternate source	ICMR Highest desirable level	ICMR Maximum Permissible level
14.	Sulphate (mg/L)	250.0	200.0	400.0	200.0	400.0
15.	Nitrate (mg/L)	45.0	45.0	45.0	20	***
16.	Phosphate (mg/L)	0.1	--	--	--	--
17.	Ammonia (mg/L)	0.5	--	--	--	--
18.	Calcium (mg/L)	100	75.0	200.0	75.0	200.0
19.	Magnesium (mg/L)	30	30.0	100.0	<50 if sulphate 200	--
20.	Chromium (mg/L)	0.05	0.05	No relaxation	--	--
21.	Iron (mg/L)	<0.3	0.3	1.0	0.1	1.0
22.	Mercury (mg/L)	0.001	0.001	No relaxation	--	0.001
23.	Colliform Cells/ 100ml	1	10	--	--	10
24.	Residual Chlorine (mg/L)	--	0.2	--	--	--
25.	Total alkalinity (mg/L)	--	200	600	--	--
26.	Chromium (mg/L)	--	0.05	No relaxation	--	--

(*) WHO [5]; Maximum permissible value = 500 mg/l on $CaCo_3$ scale

(**) [8]; Dissolved solids relaxable upto 3000 mg/l in case where alternate sources are not available within reach.

(***) [8]; more information is required to prescribe a value but in no circumstances should exceed 100 mg No_3/l.

As far as the colliform cells per 100 ml results are concerned, the water being supplied by municipal committee is quite good but the water being supplied from other reservoirs (i.e. sample no. 2, 3 and sample No.4) is expected to contain higher amounts of organic matter, especially the colliform group of microorganisms. So it is not good for the human health and a proper treatment of these water supplies is immediately desirable [7]

It is clear that the water being supplied by municipal committee is comparatively better than the water being supplied from the other reservoirs in the campus of Gurukula Kangri University. Particularly the Gaushala resevoir tank requires thorough cleansing. Further the distribution pipe lines should be repaired or replaced to avoid the leakage of soil or organic matter into the water which is definetely responsible for higher value of colliform cells in these water supplies.

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Journal of Natural & Physical Sciences Vol. 11 (1997) 87-94

ROUND-OFF STABILITY OF ITERATION PROCEDUERS FOR OPERATORS IN b-METRIC SPACES

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(Received 3.12.1997)

ABSTRACT

A.M. Ostrowski's type stability theorems of iteration procedures for Banach operators in generalized metric spaces, so-called *b*-metric spaces, are presented.

Mathematics Subject Classifications (1991) : 47H10, 47H15, 65D15.

Keywords : Stable iteration, Banach contraction, fixed point *of operators*, *b*-metric spaces.

INTRODUCTION

Consider a metric space (X, d) and a mapping $T : X \to X$. The concept of iteration procedure for finding a fixed point of T is very important and fruitfull in many areas of mathematics. However, in computation, an approximate sequence $\{y_n\}$ is used in place of a sequence of successive approximations. This leads to an idea of a fixed point iteration procedure stable with respect to T. The first stability result in metric spaces has been presented by A.M.

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Ostrowski [4]. His theorem has been extended by many mathematicians to various classes of operators (see e.g.[3], [5]-[8]).

It this paper we discuss the problem of stability of iteration procedure for operators in so called "*b-metric spaces*".

PRELIMINARY RESULTS

Now we introduce a generalization of ordinary metric space (see Czerwik [2]):

Definition 1 : Let X be a set and let $s \geq 1$ be a given real number. A function $d : X \times X \to R_+$ (the set of nonnegative real numbers) is said to be a b-metric iff for all $x, y, z \in X$ the following conditions are fullfield :

$$d(x,y) = 0 \text{ iff } x=y, \tag{1}$$

$$d(x,y) = d(y, x), \tag{2}$$

$$d(x,z) \leq s\,[d(x,y)+d(y,z)]. \tag{3}$$

Then (X,d) is called a b-metric space.

In [1] *S*. Czerwik has proved the following result about the existence and uniquenness of fixed points of operators of Banach type in b-metric spaces.

For $T : X \to X$ we denote by T^n the n-th iterate of T.

Theorem 1 : ([1]) let (X,d) be a complete b-metric space and let $T : X \to X$ satisfy

$$d[Tx, Ty] \leq \emptyset[d(x,y)],\ x, y \in X, \tag{4}$$

where $\emptyset : R_+ \to R_+$ is an increasing function such that $\lim_{n\to\infty} \emptyset^n(t) = 0$ for each fixed $t > 0$. Then T has exactly one fixed point u (*i.e.* $Tu=u$) and

$$\lim_{n\to\infty} d[T^n x, u] = 0 \text{ for each } x \in X. \tag{5}$$

Remark 1 : Theorem 1 has been proved for $s = 2$, but the proof remains unchanged for any $s \geq 1$.

First we prove the following.

Lemma : Let $\{\varepsilon_n\}$ be a sequence of nonnegative real numbers. Then

$$\lim_{n\to\infty} \varepsilon_n = 0 \quad \text{iff} \quad \lim_{n\to\infty} S_n = 0 \tag{6}$$

where

$$S_n = \sum_{r=0}^{n} \alpha^{n-r} \varepsilon_r \tag{7}$$

and

$$0 \leq \alpha < 1. \tag{8}$$

Proof : First assume that $\lim\limits_{n\to\infty} S_n \to 0$. Then since $0 \leq \varepsilon_n \leq S_n$, we get $\lim\limits_{n\to\infty} \varepsilon_n = 0$. Now, suppose $\varepsilon_n \to 0$ as $n \to \infty$. Take arbitrary $\varepsilon > 0$. We can consider $\alpha > 0$. Put $\beta = \alpha^{-1}$. There exists natural m such that $\varepsilon_n < \varepsilon$ for all $n > m$. Now we get the inequalities (for $n = m+k$, $k \in N$)

$$S_n = \alpha^n[\varepsilon_0 + \beta\varepsilon_1 + \ldots + \beta^n \varepsilon_n] = \alpha^n[(\varepsilon_0 + \ldots + \beta^m \varepsilon_m) + (\beta^{m+1}\varepsilon_{m+1} + \ldots + \beta^{m+k}\varepsilon_{m+k})]$$

$$\leq \alpha^n(\varepsilon_0 + \ldots + \beta^m \varepsilon_m) + \varepsilon\alpha^n(\beta^{m+1} + \ldots + \beta^{m+k}) \leq \alpha^n(\varepsilon_0 + \ldots + \beta^m \varepsilon_m) + \varepsilon(1-\alpha)^{-1}.$$

Therefore, there exists $n_0 \geq m$ such that for $n > n_0$ we have

$$S_n \leq \varepsilon + \varepsilon(1-\alpha)^{-1}.$$

Since $S_n \geq 0$ and $\varepsilon > 0$ is arbitrary, $\lim\limits_{n\to\infty} S_n = 0$.

This completes the proof.

STABILITY OF OPERATORS

Let X be a b-metric space and $T : X \longrightarrow X$ an operator. For any $x_0 \in X$, we define (Picard iterative procedure)

$$x_{n+1} = Tx_n,\ n = 0, 1, 2, \ldots. \tag{9}$$

Let the sequence $\{x_n\}$ be convergent to a fixed point u of T. Let $\{y_n\}$ be an arbitrary sequence of elements of X and set $\varepsilon_n = d(y_{n+1}, Ty_n)$, $n = 0, 1, 2, \ldots$.

The iteration process (9) is said to be T-stable or stable with respect to T (cf. [3]) provided that $\lim\limits_{n\to\infty} \varepsilon_n = 0$ implies that $\lim\limits_{n\to\infty} y_n = 0$.

Now we present the following result.

Theorem 2 : Let X be a complete b-metric space and $T : X \longrightarrow X$ satisfy

$$d[Tx, Ty] \leq gd(x, y) \tag{10}$$

for all $x, y \in X$ where

$$\alpha := sg < 1. \tag{11}$$

Let x_0 be an arbitrary point in X and $\{x_n\}$ a sequence of iterates of T given by (9). Let $\{y_n\}^{\infty}_{n=0}$ be a sequence in X, and set $\varepsilon_n = d[y_{n+1}, Ty_n]$, $n = 0, 1, 2, \ldots$.

Then

$$\lim_{n\to\infty} x_n = u = Tu, \tag{12}$$

$$d(u, y_{n+1}) \leq sd(u, x_{n+1}) + s\alpha^{n+1} d(x_0, y_0) + s^2 \sum_{r=0}^{n} \alpha^{n-r} \varepsilon_r. \tag{13}$$

Moreover,

$$\lim_{n\to\infty} y_n = u \quad \text{iff} \quad \lim_{n\to\infty} \varepsilon_n = 0. \tag{14}$$

Proof : From Theorem 1 for $\phi(t) = gt$ we get that T has exactly one fixed point $u \in X$, i.e. $Tu = u$ and the sequence $\{x_n\}$ has the limit u, which means that (12) holds true.

Now, in view of (10) and the traingle inequality (3) for b-metric, we obtain for nonnegative integer n,

$$d(x_{n+1}, y_{n+1}) = d\,(Tx_n, y_{n+1}) \le sd(Tx_n, Ty_n) + sd(Ty_n, y_{n+1}) \le sgd(x_n, y_n) + s\varepsilon_n$$

$$\le sg[sgd\,(x_{n-1}, y_{n-1}) + s\,\varepsilon_{n-1}] + s\,\varepsilon_n \le \alpha^2 d(x_{n-1}, y_{n-1}) + s[\alpha_{n-1} + \varepsilon_n].$$

Consequently, by induction principle, we get the inequality

$$d(x_{n+1}, y_{n+1}) \le \alpha^{n+1} d(x_0, y_0) + s \sum_{r=0}^{n} \alpha^{n-r} \varepsilon_r.$$

Therefore, we have

$$d(u, y_{n+1}) \le sd(u, x_{n+1}) + sd(x_{n+1}, y_{n+1}) \le sd(u, x_{n+1}) + s\alpha^{n+1} d\,(x_0, y_0) + s^2 \sum_{r=0}^{n} \alpha^{n-r} \varepsilon_r$$

i.e. the estimation relation (13).

To prove (14), assume that $y_n \to u$ as $n \to \infty$. Then we obtain

$$\varepsilon_n = d(y_{n+1}, Ty_n) \le sd(y_{n+1}, u) + sd(u, Ty_n) \le sd(y_{n+1}, u) + s[\,sd(u, Tu) + sd(\,Tu, Ty_n)] \le sd(y_{n+1}, u) + s^2 gd(u, y_n).$$

Therefore the assumption $y_n \to u$ as $n \to \infty$ implies $\varepsilon_n \to 0$ as $n \to \infty$.

Conversely, let $\varepsilon_n \to 0$ as $n \to \infty$. Since $x_n \to u$ as $n \to \infty$ and $0 \leq \alpha = sg < 1$, we see that the first two terms of the right hand side of (13) vanish in the limit.

In view of (6) of Lemma the third term of (13) has the limit zero. This completes the proof.

In the next theorem instead of condition (11), we consider an appropiate iterate of the operator T.

Theorem 3 : Let X be a complete b-metric space and let $T{:}X \to X$ satisfy.

$$d(Tx, Ty) \leq g d(x, y),\ x, y \in X \text{ and } 0 \leq g < 1. \tag{15}$$

Let m be a natural number such that

$$\beta : g^m s < 1 \tag{16}$$

and set $F := T^m$.

For $x_0 X$, let $x_{n+1} = F x_n$, $n = 0, 1, 2, \ldots$ let $\{y_n\}^{\infty}_{n=0}$ be a sequence in X and $\varepsilon^*_n = d(y_{n+1}, F y_n)$, $n = 0, 1, 2, \ldots$. Then T and F have exactly one common fixed point u and $u = \lim\limits_{n \to \infty} x_n$.

Moreover,

$$d(u, y_{n+1}) \leq s d(u, x_{n+1}) + s\beta^{n+1} d(x_0, y_0) + s^2 \sum_{r=0}^{n} \beta^{n-r} \varepsilon^*_r, \tag{17}$$

$$\lim_{n \to \infty} y_n = u \quad \text{if} \quad \lim_{n \to \infty} \varepsilon^*_n = 0. \tag{18}$$

Proof : Take $\phi(t) = gt$, $t > 0$, then from Theorem 1 it follows that T has exactly one fixed point $u \in X$. Now, we have for $x, y \in X$,

$$d[T^m x, T^m y] \leq g d[T^{m-1} x, T^{m-1} y] \leq g^2 d[T^{m-2} x, T^{m-2} y] \leq \ldots \leq g^m d(x, y).$$

This means that

$d(Fx, Fy) \leq g^m d(x, y)$ for all $x,\ u \in X$. Again, from Theorem 1 for $\phi(t) = g^m t$, $t > 0$, we see that F has exactly one fixed point $w \in X$ and $w = \lim\limits_{n \to \infty} x_n$. We show that $u = w$. Indeed, $u = Tu = T^2 u = \ldots = T^n u = Fu$. T and F has only one fixed point $u = w$.

The statements (17) and (18) are easily verified as in the proof of Theorem 2.

Remark 2 : Under the assumptions of Theorem 3, $\lim\limits_{n \to \infty} \varepsilon^*_n = 0$ implies

$$\lim_{n \to \infty} \varepsilon_n = \lim_{n \to \infty} d(y_{n+1}, Ty_n) = 0.$$

Indeed, since $Tu = u$, we have

$$\varepsilon_n = d(y_{n+1}, Ty_n) \leq sd(y_{n+1}, Tu) + sd(Tu, Ty_n) \leq sd(y_{n+1}, u) + sgd(u, y_n) \to 0$$

as $n \to \infty$.

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Journal of Natural & Physical Sciences Vol. 11 (1997) 95-106

PROPAGATION OF CORRELATIONS IN DECAY PROCESS OF HYDRODYNAMIC AND MHD TURBULENCE BEFORE THE FINAL PERIOD

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(Received 8.12.1997)

ABSTRACT

The present paper reports a theoretical calculation of correlation functions. The decay of MHD turbulence at times before the final period is discussed.

Keywords : MHD turbulence, correlation functions. Decay process.

INTRODUCTION

The manner in which a fluid in turbulent motion transports and modifies the initial shape of physical quantities is of considerable importance in many fields. If we agree to call the motion of physical quantities in turbulent fluids dispersion, we may cite many example of this process. the motion of clouds in the sky, smoke from a smokestack, salt concentration in the sea, electron density in the ionosphere, staruparticles in galactic clouds, and temperature in interstellar regions are but a few example of the turbulent dispersion phenomenon. In these years many authors studies the temperative dispersion.

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T. Dixit [2] consider the effect of corriolis force on the thermal Decay process of M H D turbulence. Sarkar and Kishore [1] studied the decay of M H D turbulence at times before the final period. In this paper we have discussed the decay of M H D dusty turbulence before the final period.

Two Point Correlaion and special Eqations

Let u_i denote the component of velocity at point $p(x_i)$ then

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_k}(u_i u_k - h_i h_k) = -\frac{\partial w}{a x_i} + \nu v^2{}_x \cdot u_j + f(v_i - u_i) \quad (3.1)$$

where $w = P/\rho + \frac{1}{2}\ h^2$ is total MHD pressure, $p(x,t)$ is the hydromagnetic pressure.

The corresponding equation for the point p' will be

$$\frac{\partial u'_j}{\partial t} + \frac{\partial}{\partial x'_k}(u'_i u'_k - h'_i h'_k) = \frac{\partial w'}{\partial x'_j} + \nu v^2{}_{x'} \cdot u'_j + f(v'_j - u'_j) \quad (3.2)$$

Multiplying (3.1) by u'_j and (3.2) by u_i, adding and taking ensemble average, we get

$$\frac{\partial}{\partial t}\ \overline{u_i u_j} + \frac{\partial}{\partial x_k}\ \overline{(u_i u_k u'_j - h_i h_k u'_j)} + \frac{\partial}{\partial x'_k}\overline{(u'_i u'_k u_i - h'_i h'_k u_i)}$$

$$= -\ \frac{\partial}{\partial x'_j}\ \overline{w' u_i} - \ \frac{\partial}{\partial x'_i}\ \overline{w u'_j} + \nu v^2{}_x\ \overline{u'_j u_i} + \nu v^2_x\ \overline{u_i u'_j} + \mathrm{f}(\ u_i u'_j - u_i u'_j$$

$$+\ \overline{u'_j u_i} - \overline{u_i u'_j} \qquad \ldots(3.3)$$

using the transformation

$$\frac{\partial}{\partial r_k} = - \frac{\partial}{\partial x_k} = \frac{\partial}{\partial x'_k}$$

$$r_i = x'_i - x_i, \quad \nu^2_{x'} = \nu^2$$

and relations [3]

$$\overline{u_k \; h_i \; h'_j} = - \overline{h_i \; u'_k h'_j}\,, \quad \overline{u_i \; u_j u'_k} = - \overline{u'_i \; u'_j u_k}$$

and

$$\overline{h_i \; h'_j h'_k} = - \overline{u_i \; h_k h'_j}\,, \quad \overline{h_i \; h_j u'_k} = - \overline{h'_i \; h'_j u_k}$$

We consider that dust particles are non conducting and hence $\overline{h_i v'_j}$ = $h'_j \, u'_i$ = 0. We also assume that the instantaneous velocities at the point remain unaffected by the dust particles of the other point, i.e.

$$\overline{u_i \; v'_j} = \overline{u'_j \; v_i} = 0.$$

Also $\overline{w' \, u_i}$ and $\overline{w' \, v'_j}$ vanish (see 37)

Hence equation (3.3) can be written as

$$\frac{\partial}{\partial t} \overline{u_i \, u'_j} - 2 \frac{\partial}{\partial r_k} (\overline{u_i \, u_k u'_j} - \overline{h_i \, h_k h'_j})$$

$$= 2 \, \nu \nu^2 \; \overline{u_i \, u'_j} - 2f \; \overline{u_i \; u'_j} \qquad ..(3.4)$$

Induction equation of a magnetic field at point p is

$$\frac{\partial h_i}{\partial t} + u_k \frac{\partial h_i}{\partial x_k} - h_k \frac{\partial h_i}{\partial x_k} = \frac{\nu}{P_m} \frac{\partial^2 h'_j}{\partial x_k \, \partial x_k} \qquad ..(3.5)$$

where $P_M = \frac{\nu}{\lambda}$ is magnetic Prandtl number, is the magnetic diffusivity, $h_i(x,t)$ the magnetic field fluctuation, uk(x,t) is the turbulent velocity, ν is kinematic viscosity.

The corresponding equation for the point p' will be

$$\frac{\partial h_j}{\partial t} + u'_k \frac{\partial h'_j}{\partial x_k} - h'_i \frac{\partial u'_j}{\partial x'_k} = \frac{\nu}{P_m} \frac{\partial^2 h'_j}{\partial x'_k \, \partial x'_k} \qquad ..(3.6)$$

Multiplying equation (3.5) by h'_j and (3.6) by h_i adding and taking ensemble average, we get

$$\frac{\partial}{\partial t} \overline{h_i h'_j} + \frac{\partial}{\partial x_k} \overline{u_k h_i h'_j} + \frac{\partial}{\partial x'_k} \overline{u'_k h_i h'_j}$$

$$- \frac{\partial}{\partial x_k} \overline{u_i h_k h'_j} - \frac{\partial}{\partial x_k} (\overline{u'_j h_i h_k})$$

$$= \frac{\nu}{P_M} \left(\frac{\partial^2 \overline{h_i h'_j}}{\partial x_k \, \partial x_k} + \frac{\partial^2 \overline{h_i h'_k}}{\partial x_k \, \partial x'_k} \right)$$

or $$\frac{\partial}{\partial t} \overline{h_i j'_j} + 2 \frac{\partial}{\partial r_k} u'_k h_i h'_j - \frac{\partial}{\partial r_k} \overline{u_i u_k h_j}$$

$$= 2 \frac{\nu}{P_M} \frac{\partial^2}{\partial r_k \, \partial r_k} + \overline{h_i j'_j} \qquad ..(3.7)$$

Let

$$\overline{h_i h'_j (r)} = \int_{-\infty}^{\infty} \psi_i \psi'_j (k) \exp (ik.r) \, dk.$$

$$\overline{u_k h_k h'_k (r)} = \int_{-\infty}^{\infty} \alpha_i \psi_k \psi'_j (k) \exp (ik.r) \, dk.$$

Interchanging the subscripts *i* and *j* and then interchange the points *P* and *P'* we have

$$\overline{u'_k h_i h'_j (r)} = \overline{u_k h_i h'_j (-r)}$$

$$= \int_{-\infty}^{\infty} \alpha_k \psi_i \psi'_j (-k) \ \exp \ (ik.r)\, dk.$$

Therefore (3.7) can be written as

$$\frac{\partial}{\partial t} \overline{\psi_i \psi'_j (k)} + 2 \frac{\nu}{P_M} k^2 \overline{\psi_i \psi'_j (k)} = 2i \ k_k \ [\overline{\alpha_i \psi_k \psi_j (k)} - \overline{\alpha_k \psi_i \psi'_j (-k)}] \qquad ...(3.8)$$

The tensor equation (3.8) becomes a scalar equation by contraction of indicies *i* and *j*

$$\frac{\partial}{\partial t} \overline{\psi \psi'_j (k)} + 2 \frac{\nu}{P_M} k^2 \overline{\psi_i \psi'_j (k)} = 2i \ k_k [\overline{\alpha_i \psi_k \psi'_j (k)} - \overline{\alpha_k \psi_i \psi'_j (-k)}] \qquad ...(3.9)$$

THREE POINT CORRELATION AND EQUATIONS

We take momentum equation at the point *P* and the induction equations of magnetic field fluctuations at *p'* and *p'*

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_1}{\partial x_k} - h_k \frac{\partial h_1}{\partial x_k} = - \frac{\partial w}{\partial x_1} + \nu \frac{\partial^2 u_1}{\partial x_k \partial x_k} + f(v_1 - u_1) \qquad ...(4.1)$$

$$\frac{\partial h'_i}{\partial t} + u'_k \frac{\partial h'_i}{\partial x'_k} = - h'_k \frac{\partial u'_i}{\partial x'_k} = \frac{\nu}{P_M} \frac{\partial^2 h'_i}{\partial x'_k \partial x'_k} \qquad ...(4.2)$$

and $\frac{}{\partial t} + u'_k{}' \frac{}{\partial x'_k{}'} - h'_k{}' \frac{}{\partial x'_k{}'} = \frac{}{P_M} \frac{}{\partial x'_k{}' \partial x'_k{}'}$ ...(4.3)

Multiplying (4.1) by $h'_i \ h'_j{}'$ (4.2) by $u_1 \ h'_j{}'$ and (4.3) by $u_1 \ h'_i$ adding the three equations and taking averages, we obtain

$$\frac{\partial}{\partial t} \overline{u_1 h'_i h''_j} + \frac{\partial}{\partial x_k} \overline{u_1 u_k h'_j \ h'_j{}'} - \frac{\partial}{\partial x_k} \overline{h_1 h_k h'_i \ h'_j{}'}$$

$$+ \frac{\partial}{\partial x'_k} \overline{u_1 u'_k h'_i \ h'_j{}'} - \frac{\partial}{\partial x'_k} \overline{u_1 u'_i h'_k \ h'_j{}'} + \frac{\partial}{\partial x'_k{}'} \overline{u_1 u'_k{}' h'_i \ h'_j{}'}$$

$$- \frac{\partial}{\partial x'_k{}'} \overline{u_1 u'_j{}' h'_i \ h'_k{}'}$$

$$= - \frac{\partial}{\partial x'_1} \overline{u_i h'_i h'_j{}'} + \frac{\partial^2 u_1 h'_i h'_j{}'}{\partial x_k \ \partial x_k} + \frac{\nu}{P_M} \left[\frac{\partial^2 \overline{u_1 h'_i h'_j{}'}}{\partial x'_k \ \partial x'_k}\right.$$

$$\left. + \frac{\partial^2 \overline{u_1 h'_i h'_j{}'}}{\partial x'_k{}' \ \partial x'_k{}'}\right] + f \overline{(\nu_i h'_i h'_j{}'} - \overline{u_1 h'_i h'_j{}')} \qquad ...(4.4)$$

Using the transformations

$$\frac{\partial}{\partial x'_k} = \frac{\partial}{\partial r_k}, \quad \frac{\partial}{\partial x'_k{}'} = \frac{\partial}{\partial r'_k}$$

and $\frac{\partial}{\partial x_k} = -\left(\frac{\partial}{\partial r_k} + \frac{\partial}{\partial r'_k}\right)$

into equation (4.4), we get

$$\frac{\partial}{\partial t} \overline{u_1 h'_i h'_j{}'} - \frac{\nu}{P_M} \left[(1+P_M) \frac{\partial^2 \overline{u_1 h_i h'_j{}'}}{\partial r_k \ \partial r_k} (1+P_M) \frac{\partial^2 \overline{u_1 h'_i h'_j{}'}}{\partial r'_k \ \partial r'_k}\right.$$

$$+ 2P_M \quad \frac{\partial^2 \overline{u_1 \, h_i \, h'_j{}'}}{\partial r_k \quad \partial r'_k} = \frac{\partial}{\partial r_k} \overline{u_1 \, u_k h'_i h'_j{}'} + \frac{\partial}{\partial r'_k} \overline{u_1 \, u_k \, h'_i h'_j{}'}$$

$$- \frac{\partial}{\partial r_k} \overline{h_1 \, h_k \, h'_i h'_j{}'} - \frac{\partial}{\partial r'_k} \overline{h_1 \, h_k \, h'_i h'_j{}'}$$

$$- \frac{\partial}{\partial r_k} \overline{u_1 \, u'_k \, h'_i h'_j{}'} + \frac{\partial}{\partial r'_k} \overline{u_1 \, h'_k h'_j{}'}$$

$$- \frac{\partial}{\partial k'} \overline{u_1 \, u'_k{}' h'_i h'_j{}'} + \frac{\partial}{\partial r'_k} \overline{u_1 u'_j{}' h'_i h'_j{}'} + \frac{\partial}{\partial r'_1} \overline{w h'_i \, h'_j{}'}$$

$$+ \frac{\partial}{\partial r'_i} \overline{w h'_i \; h'_j{}'} \quad + \mathrm{f} \, (\overline{u_1 \, h'_i h'_j{}'} - \overline{u_1 \, h'_i h'_j{}'}) \; f \, (\overline{u_1 \, h'_i h'_j{}'} - \overline{u_1 \, h'_i h'_j{}'}) \quad \ldots(4.4)$$

Let $\overline{u_1 \, h'_i(r) \, h'_j{}'(r')} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\phi_1 \, \overline{\beta_i(k) \, \beta'_j{}'(k')} \exp [\overline{i \, (k.r+k'.r')}] \, dkdk'$

$$\overline{u_1 \, u'_k(r) h_i(r) \, h'_j{}'(r')} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_1 \quad \overline{\phi'_k(k) \, \beta'_j{}'(k') \, \beta'_j{}'(k')}$$

$$\exp \; [i \, \overline{(k.r+k'.r')}] \, dkdk'$$

$$\overline{u_1 \, u'_i(r) \, h'_k(r) h'_j{}'(r')} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\phi_i(k) \, \beta_k{}'(k) \, \beta'_j{}'(k) \, \beta'_j{}'(k')}$$

$$\exp \; [i \, (k.r+k'.r')], \, dkdk'$$

$$\overline{u_1 \, u_k h'_i(r) \, h''(r'_j)} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_1 \, \overline{\phi_k \beta_i{}' \, \beta'_j{}'(k')}$$

$$\exp \; [i \, (k.r+k,r')], \, dkdk'$$

$$\overline{h_1 \, h_k h'_i(r) \, h'_j{}'(r')} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\beta_1 \, \beta_k \beta'_i(k) \, \beta'_j{}'(k')}$$

$$\exp \, [i(k.r + k'.r')] \, dk \, dk'$$

$$\overline{wh'_i(r)\, h'^{\,\prime}_j(r')} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\gamma\, \beta_i\ (k)\ \beta'^{\,\prime}_j\ (k')}$$

$$\exp\ [i(k.r + k'.r')]\ dk\ dk'$$

$$\overline{v_1\, h'_i(r)\, h'^{\,\prime}_j(r')} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\delta_1\, \beta'_i\ (k)\ \beta'^{\,\prime}_j\ (k')}$$

$$\exp.\ [i(k.r + k'.r')]\ dk\ dk'$$

Interchange of points p' and p'' along with the subscripts i and j, results in the relations

$$\overline{u_1\, u'^{\,\prime}_k\ h'^{\,\prime}_j\ h'_i} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{u_1\, u'_k\ h'_i\ h'^{\,\prime}_j}$$

and

$$\overline{u_1\, u'^{\,\prime}_j\ h'_i\ h'^{\,\prime}_k}\ . = \overline{u_1\, u'_i\ h'_k\ h'^{\,\prime}_j}$$

By use of the above facts and relations. The equation (4.5) may be written as

$$\frac{\partial}{\partial \mathrm{t}} \overline{\phi_1\, \beta'_i\ \beta'^{\,\prime}_j} + \frac{\nu}{P_M}[(1+P_M)k^2 + (1+P_M)\ k^2\, 2P_M\, k_k\, k'_k] + f\,\frac{P_M}{\nu}\ \phi_1 \beta'_i \beta'_i$$

$$= i(k_k + k'_k)\ \ \overline{\phi_1\ \phi_k \beta'_i\ \beta'^{\,\prime}_j}\ \ —\ \ i(k_k + k'_k)\ \ \overline{\beta_1\ \beta_k\, \beta'_i\ \beta'^{\,\prime}_j}$$

$$= i(k_k + k'_k)\ \ \overline{\phi_1\ \phi_k \beta'_i\ \beta'^{\,\prime}_j}\ + \ i(k_k + k'_k)\ \ \overline{\phi_1\ \phi'_i\ \beta'_k\ \beta'^{\,\prime}_j}$$

$$+\ i(k_1 + k'_1)\ \ \gamma \overline{\beta'_i\ \beta'^{\,\prime}_j}\ + f\,(\overline{\delta_1\ \beta'_i\ \beta'^{\,\prime}_j}) \qquad \ldots.(4.6)$$

The tensor equation (4.6) can be converted to scalar equation by contraction of indices i and j given by

$$\frac{\partial}{\partial \mathrm{t}}\ \overline{\phi_1\, \beta'_i\ \beta'^{\,\prime}_j} + \frac{\nu}{P_M}\ [(1+P_M)k^2 + k'^2) + 2P_M kk' + f\ \frac{P_M}{\nu}]\ \overline{\phi_1 \beta'_i \beta'_j}$$

$$= i(k_k + k'_k)\ \overline{\phi_1\, \phi_k\, \beta'_i\ \beta'_j{}'}\ -\ i(k_k+k'_k)\ \overline{\beta_1\, \beta_k\, \beta_1\ \beta'_j{}'}$$

$$-\ i(k_k+k'_k)\ \phi_1\, \overline{\phi'_k\, \beta'_i\ \beta'_L{}'}\ +\ i(k_k + k'_k)\ \overline{\phi_1\, \phi'_i\ \beta_k\, \beta'_i{}'}$$

$$+\ \gamma\,(k_1+k'_1)\ \overline{\beta'_i\ \beta'_i{}'}\ +\ f\,\overline{(\delta_1\ \beta_i{}'\ \beta'_j{}')} \qquad ....(4.7)$$

If we take the derivative with respect to x_1 of the momentum equation (4.1) we have

$$\frac{\partial^2 w}{\partial x_1\, \partial x_1} = \frac{\partial^2}{\partial x_1\, \partial x_k}\ (u_i\, u_k - h_1\ h_k) \qquad ....(4.8)$$

Multiplying equation (4.8) by $h'_i\ h'_j{}'$ taking average and writing equation in terms of the independent variables r and r'

$$-\ \left[\frac{\partial^2}{\partial r_1\ \partial r_1} + \frac{\partial^2}{\partial r'_1\ \partial r'_1} + \frac{\partial^2}{\partial r_1\ \partial r'_1}\right]\ \overline{w\, h'_i\ h'_j{}'}$$

$$= \left[\frac{\partial^2}{\partial r_1\ \partial r_k} + \frac{\partial^2}{\partial r'_1\ \partial r'_k} + \frac{\partial^2}{\partial r_1\ \partial r'_k} + \frac{\partial^2}{\partial r'_1\ \partial r'_k}\right]$$

$$.\ [\overline{u_1\ u_k\ h'_i\ h'_j{}'}\ -\ \overline{h_1\, h_k\ h'_j{}'}] \qquad ....(4.9)$$

Taking the fourier transforms of equation (4.9) we get

$$-\ r\,\overline{\beta'_i\ \beta'_j{}'} = \frac{(k_1\, k'_k + k'_1\, k_k + k'_{k'} + k'_1\ k'_k)\ (\phi_1\, \phi_k\, \beta'_i\ \beta'_J{}'\, \beta_1\, \beta_k\, \beta'_i\ \beta'_J{}')}{k^2 + k'^2 + 2k_1\, k'_l} \qquad ....(4.10)$$

Equation (4.10) can be used to eliminate $r\,\beta'_1\ \beta'_J{}'$ from equation (4.7).

SOLUTION FOR TIMES BEFORE THE FINAL PERIOD

It is known that the equation for final period decay is obtained by considering the two point correlation after neglecting the third order correlations. To study the decay for times before the final period, the three point correlation equations are considered and the quadruple corelations are neglected. By making use of (4.10) and neglecting quadruple correlations the equation (4.7) can be written as

$$\frac{\partial}{\partial t}\ d_1 \beta'_i\, \beta'^{\,\prime}_i + \frac{\nu}{P_M}\,[(1+P_M)\ \ (k^2+k'^2) + f\frac{P_M}{\nu} + 2P_M\, kk']$$

$$\phi_1 \beta'_i\, \beta'^{\,\prime}_i\ = 0 \qquad ....(5.1)$$

Intigrating the equation (5.1) between t_0 and t with inner multiplication by k_k gives

$$k_k\, \phi_1\, \beta'_i\, \beta'^{\,\prime}_i\ = k_k\ \overline{[\phi_1\, \beta'_i\, \beta'^{\,\prime}_1]}_0$$

$$.\exp\,[-\frac{\nu'}{P_M}\,(1+P_M)\,(k^2+k'^2) + f\ \frac{P_M}{\nu} + 2P_M\, k'_k\ \cos\,\theta)\ (1+t_0)]$$

$$....(5.2)$$

where θ is the angle between k and k'.

Now by letting $r' = 0$ in fourier transforms we have

$$\overline{\alpha_i\, \Psi_k \Psi'_j\,(k)}\ = \int_{-\infty}^{\infty}\ \ \overline{\phi_i\ \beta'_i\, \beta'^{\,\prime}_i}\, dk.$$

and

$$\alpha_k\ \Psi_i\, \Psi'_i\,(-k)\ = \int_{-\infty}^{+\infty}\ \ \phi_i\ \beta'_i\,(-k)\beta'^{\,\prime}_i\ (-k')\ dk \qquad ....(5.3)$$

substituting (5.2) and (5.3) in (4.9) we have

$$\frac{\partial}{\partial t}\,\overline{\Psi_i\, \Psi'_j\,(k)}\ + 2\ \ \frac{\nu}{P_M}\ \ k^2\ \overline{\Psi_i\, \Psi'_i\,(k)} = \ 2i\ \ k_k\ [\phi_1\ \beta'_i\, \beta'^{\,\prime}_i$$

$$-\ \overline{\phi_1\ \beta'_i\,(-k)\ \ \beta''_{i_i}\,(-k')]_0}$$

$$\exp\,[-\frac{\nu}{P_M}\,(1+P_M)\,(k^2+k'^2) + f\ \frac{P_M}{\nu}\ + 2P_M\, k'_k\ \cos\,\theta)](t-t_0)\ dk' \ .... (5.4)$$

$$dk' = 2\pi k^2 \, d\,(\cos\theta)\, \mathrm{dk}' \qquad \text{....(5.5)}$$

substituing (5.5) to equation (5.4) gives

$$\frac{\partial}{\partial t} \; \overline{\Psi_i \Psi'_j(k)} + 2 \; \frac{\upsilon}{P_M} \; k^2 \; \overline{\Psi_i \Psi'_i(k)} \; = \; 2 \int_0^\infty 2\pi i k_k \, [\overline{\phi_1 \, \beta'_i \, \beta'_j}'$$

$$— \; \phi_1 \overline{\beta'_c(-k') \;\; \beta'_i{}' \;\; (-k')-}]_0 \, k'^2$$

$$\times \;\; [\exp [- \frac{\nu}{P_M} \; (t-t_o) \; [(1+P_M) + (k^2+k'^2)+P_M \, K'_k \, \cos\theta$$

$$+ f \frac{P_M}{\upsilon}] \;\; d\,(\cos\theta) \,]\, dk' \qquad \text{.... (5.6)}$$

Again following [1] we assume that

$$ik_k \, [\overline{\phi_1 \, \beta'_i \;\; \beta'_i}' \;\; — \;\; \phi_1 \overline{\beta'_i(-k)\beta''_i(-k')}]_o$$

$$= \; — \;\; \frac{\xi_o}{(2\pi)^2} \;\; (k^2k'^4 — k^4k'^2) \qquad \text{.... (5.7)}$$

Where ξ_o is a constant depending on the initial conditions. From (5.6) and (5.7) we get after integrating with respect to cos θ)

$$\frac{\partial}{\partial t} \;\; 2\pi \; \overline{\Psi_i \Psi'_j(k)} + 2 \;\; \frac{\nu}{P_M} \;\; k^2 \;\; (2\pi \; \overline{\Psi_i \Psi'_i(k)})$$

$$= \; — \;\; \frac{\xi_o}{\nu(t-t_0)} \; \int_o^\infty (k \;\; k'^s — k^2k'^3)$$

$$[\exp [- \frac{\nu}{P_M} \; (t-t_o) \; [(1+P_M) \, . \, (k^2+k'^2)+ f \;\; \frac{P_M}{\nu} \; — 2P_M \, kk']]$$

$$— \exp\left[-\frac{\nu}{P_M}(t-t_o)\left[(1+P_M)+(k^2+k'^2)+f\frac{P_M}{\nu}+2P_M kk'\right]\right]dk \quad ..(5.8)$$

Multiplying both sides by k², we get

$$\frac{\partial H}{\partial t}+\frac{2\nu\, k^2}{P_M}H = G. \quad .... (5.9)$$

Where $H = 2\nu\, k^2\ \overline{\psi_i\, \psi'_i\,(k)}$ is the magnetic energy spectrum function and G is the energy transfer terms given by

$$G = —\frac{\xi o}{(t-t_o)}\int_0^\infty (k^{3}*k'^{5}-k^5k'^3)\Big[\exp\Big(\frac{\nu}{P_M}(t-t_o)\ [1+P_M)\ (k^2+k'^2)$$

$$+f\frac{P_M}{\nu} — 2P_M\, kk'\Big]\Big) —\exp\Big(\frac{-\nu}{P_M}(t-t_o)\ [(1+P_M)\ (k^2+k'^2)$$

$$+\ 2P_M\, kk'\Big]\Big)+f\frac{P_M}{\nu}\Big]\ dk' \quad ....(5.10)$$

Following [1] if we put the value of G is (5.9) we can find H.

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Journal of Natural & Physical Sciences Vol. 11 (1997) 107-112

MOTION OF PARTICLES SUSPENDED IN TURBULENT FLOW

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(Received 8.12.1997)

ABSTRACT

In this paper we have derived the expression for velocity correlation in terms of defining scalars.

Keywords : Isotropic turbulence Reynolds number.

INTRODUCTION

Theoretically very little is known about the fluid dynamics of a mixture of discrete particles of arbitrary size and concentration with a fluid, where both are in turbulent motion. For high concentrations of very fine particles, the mixture may be considered a homogeneous suspension and may be treated is a non Newtonian fluid with certain over-all flow features. For bigger particles, such as grains and sand, Bagnold has studied stress-strain relationships of the mixture, in nearly turbulent motion. An extensive theoretical study has been made by Corrsin [1]. Saffman [2] derived an equation that describes the motion of a fluid containing small dust particles.

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Dixit [3] discussed the rate of change of vorticity covariance in MHD dusty turbulence. In this paper we consider the motion of solid particles with respect to the turbulent gas in which they are suspended. the result obtained can also be applied to the motion of solid particles, underformed droplets and bubbles in a liquid and to the motion of small liquid droplets in a turbulent gas. Here we will consider only the case in which the specific volume occupied by the particles is small compared to unity. The turbulent flow is assumed to be locally homogenous, isotropic and stationary and dimension R of the suspended particles is assumed to be so small that the Reynolds number for the relative motion of the gas is small compared with unity.

EQUATION OF MOTION

The equation of motion of particles was obtained by Corsin and Lumely [1]. This equation relates the velocity W_i of a particle in a gas to the velocity of motion U_i of the gas. In general from it is a non linear particle differential equation. It can however be transformed into a linear integro-differential equation with the time as the only independent variable if

$$\frac{R^2}{\nu}\frac{\partial U}{\partial x} << 1, \frac{W}{\nu}\frac{\partial U}{\partial x} \Big/ \frac{\partial^2 U}{\partial x^2} >>> 1.$$

where ν is the kinematic viscosity of the gas, let U_λ and W_λ be the velocity and frequency of pulsation with dimensions. Using the fact that the particle velocity is close to the velocity U_L of the largest oscillations of the gas, which are of a scale L of order of the dimensions of the system, we have

$$\frac{W}{\nu}\frac{\partial U}{\partial x} \Big/ \frac{\partial^2 U}{\partial x^2} \sim \frac{W}{\nu}\frac{U_\lambda}{\lambda} \Big/ \frac{U_\lambda}{\lambda^2} \sim \frac{U_L\lambda}{\nu} > \frac{U_L}{\nu}\frac{1}{\mathrm{L}}$$
$$\sim R_e L^{1/4} >> 1 .$$

$$\frac{R^2}{\nu}\ \frac{\partial U}{\partial x} \sim \frac{R^2}{\nu}\ \frac{U_\lambda}{\lambda} \sim \frac{R^2}{\nu}\ w_\lambda < \frac{R^2}{\nu}\ \frac{\nu}{l^2}\ \frac{R^2}{l^2} < 1.$$

Hence, if $R<1$ the equation of steady relative particle motion is

$$\frac{dV_i}{dt} = (\alpha\text{-}1)\frac{dU_i}{dt} - \beta V_i - \sqrt{\frac{3\alpha\beta}{\pi}} \int_{-\infty}^{t} \frac{dV_i}{dt}\ \frac{dT}{\sqrt{t-T}} + g_i \qquad (2.1)$$

$V_i = W_i\text{-}U_i$ is the velocity of a particle with respect to the gas,

$\alpha = \dfrac{3\rho_o}{2\rho+\rho_o}$, where ρ_o is the gas density and ρ is the particle density, g_i is the acceleraton of freefall, and $\beta = \dfrac{k_1\alpha\nu}{R^2}$ is the Characteristic frequency determing the motion of the particles in the gas and is equal to the reciprocal of the time required for the gas flowing past a particle to alter its flow state. The quantities k_i are numerical constants of order unity. Since (2.1) as linear, the velocity V_i is the superposition of two independent quantities : the velocity due to gravity and the velocity related to the motion of the gas. Since here we are interested inthe particle motion caused by the movement of the gas, term g_i can be omitted from (2.1). Under our assumption $R<1$, (2.1) can be further siplified. It can be shown that the term involving the integral is small compared to the remaining $\dfrac{w_\lambda R^2}{\nu} << 1$ for all frequencies of the turbulence spectrum .

In fact

$$w_\lambda \frac{R^2}{\nu} < \frac{w_i R^2}{\nu} \sim \frac{R^2}{l^2}$$

Hence, under our assumptions the equation for the relative motion of a particle in the gas is

$$\frac{dV_i}{dt} = (\alpha-1)\,\frac{dU_i}{dt} - \beta B_i \tag{2.2}$$

In turbulent flow the velocities U_i and V_i are random functions of the time. To determine the properties of the motion of the particles with respect to the gas we must know the relation between the correlation of U_i and correlation of V_i. Equation (2.2) can be written as

$$\frac{dx_i}{dt} = -\beta V_i \tag{2.3}$$

Where $x_i = V_i = (\alpha-1)\,U_i$

DISCUSSION OF THE PROBLEM

Let x'_i and V'_i be the values of the variables at time t at point $P(x'_i)$ and X''_j and V''_j be the values of variables at point $P(x''_j)$ then

$$\frac{dx'_i}{dt} = -\beta V'_i \tag{2.3}$$

and $$\frac{dx''_i}{dt} = -\beta V''_j \tag{3.2}$$

From Equation (3.1) and (3.2) we have

$$\frac{d}{dt}\,\overline{x'_i \;\; x''} = \beta^2\,\overline{V'_i \;\; V''_j} \tag{3.4}$$

Let $\overline{x'_i \; x''_j} = P_{ij}$

and $\overline{v'_i \; v''_j} = R_{ij}$

P_{ij} and R_{ij} being isotropic tensors can be written in the from

$$P_{ij} = \alpha(r,t)\, \zeta_i \zeta_j + \beta\,(r,t)\, \zeta_{ij} \tag{3.5}$$

and $$R_{ij} = \frac{R}{r} \zeta_i \zeta_j - (r,R' + 2R)\, \delta_{ij} \tag{3.6}$$

Putting these in (3.4) me have

$$\frac{\partial}{\partial t}\alpha(r,t) = \beta^2 \frac{R}{y}$$

and $$\frac{\partial}{\partial \mathrm{t}}\beta(r,t) = -\beta^2\,(rR' + 2R) \tag{3.8}$$

where $\zeta_{\mathrm{j}} = x''_{\mathrm{i}} - x'_{\mathrm{i}}$

The equations (3.7) and (3.8) give the expression for the rate of change of velocity covariance of particles in terms of defining scalars.

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Journal of Natural & Physical Sciences Vol. 11 (1997) 113-120

INFLUENCE OF AIR POLLUTION ON WEATHER THROUGH SOLAR ACTIVITY

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(Received 15.01.1998)

ABSTRACT

Solar activity affects the ionization in the atmosphere. The atmospheric ions are attached to aerosol particles. These heavy charged particles change the atmospheric conductivity due to change in ion density and mobility. These ions also affect nucleation, formation of rain and lightning activity. Thus these parameters are likely to be strongly correlated with ionization in the atmosphere. Therefore pollution particles should enhanced the type of correlation of precipitation and lightning activity with solar activity. On the basis of above results, we have described the atmospheric processes blockwise, putting one block for each mechanism. Each block may be treated as a system with occasional feedback and dissipation. Using the standard technique of system analysis, one can find the output of each block in exact mathematical terms.

INTRODUCTION

Some atmospheric parameters like precipitation, snowfall, atmospheric electric field, lightning activity, surface temperature, pressure and wind are known to correlate with solar cycles [4].

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Radiations from the sun is the dominant source of energy input to the earth's atmosphere. Sun is not only the primary source of all forms of energies on earth, it is also responsible for wheather phenomenon on our planet. The changes occuring in sun are deemed to affect the weather. A direct link between solar variability and weather changes is indeed a prime factor. The average incoming electromagnetic radiation received from the sun at the top of the atmosphere over all wavelengths is 1353 watt/m^2 [4]. This energy is really constant with time and therefore, it is known as 'Solar Constant'.

It is generally beleived that solar activity modulates the flux of galactic cosmic radiation [2]. This solar modulation takes place as the galactic cosmic rays propagate through the region around the sun containing the interplanetary medium. This interplanetary medium controls the primary galactic cosmic radiations and thereby, the flux of secondary cosmic radiations are generated in the stratosphere [9]. The later is generally responsible for atmospheric ionization.

Cosmic rays are known to create enhanced ionization in the atmosphere. This would increase the conductivity. Increased conductivity decreases atmospheric electrical field. The atmospheric ionization contributes significantly to the charge separation process in convective clouds. During the formation of convective clouds, ions present in the atmosphere are carried with the convection to higher levels where it is surrounded by lower ion density. Hence, it creates stronger field to attract ambient opposite ions to migrate towards cloud boundary. These ions are transported to bottom of the cloud with downward air movements. Naturally, the higher ionization at lower atmosphere will result in stronger charge separation giving enhanced lightning activity [14].

The effect of sun on atmospheric electricity was observed for the first time by Watson [15]. Stringfellow [13] found that the 5 year running mean of annual mean sunspot numbers and the annual index of lightning were in phase for the period 1930 to 1970. Markson [8] explained this correlation in terms of ionizing radiation from sun. An increase in the ionization decreases the atmospheric resistivity, thereby causing decrease in the atmospheric electric field. The change in atmospheric electric field affects the process of charge separation and precipitation.

LATITUDE AND ALTITUDE DEPENDENCE OF COSMIC RAY IONIZATION

The cosmic ray ionization depends upon both the latitudes and altitudes. Manzano and Winckler [7], Shea et al. [12], Nehar [10], Kent and Pomerantz [5], Lockwood et al. [6] and Agarwal [1] studied the cosmic ray ionization with latitudes. Ion production rate by galactic cosmic rays increases from equator to about 50^0-60^0 geomagnetic latitude, where it breaks to form a "knee" [10]. Polweard of the knee, the ion production rate is constant because of decreased magnetic rigidity effects in the nearly vertical geomagnetic field. Nehar [10] proposed that rigidity effects vary through the solar cycle. Ionization by cosmic rays also depends on altitudes. Maximum ion production takes place at altitudes of about 12 to 20 km. The height dependence of cosmic ray ionization has been given by Gish [3] for magnetic latitudes 3^0N and 51^0N. Agarwal [1] showed that the ground level ionization at all latitudes is mainly due to galactic cosmic rays [GCR]. However, at high latitudes the main source of ionization around 20 km and above is due to solar cosmic rays [SCR]. Shea et al. [12] explains the phenomena is terms of Forbush decrease where increased flux of SCR decrease the GCR intensity due to electrostatic shielding of the later.

According to Shea et al.[12] the solar wind protons at high latitudes are dominant over the galactic cosmic rays. Upto about 30, the ion pair production rate is approximately constant, after which it increases very fast. Nehar [10, 11] reported that cosmic ray intensity is minimum at the equator and increases by about two orders of magnitude until 60^0 geomagnetic latitude. After this it becomes constant. From the above we conclude that at low latitudes the atmospheric ionization is mainly due to GCR while at high latitudes it is due to SCR. At mid latitudes from 30^0 to 50^0 contribution is from both sides and it is highly fluctuating.

TIME VARIATION OF LIGHTNING ACTIVITY

Let S_0 be the solar visible and infra-red energy input to the atmosphere with cosmic ray energy E in presence of albedo [which transfers AS_0 energy to warming and reflects back $(1\text{-}A)\ S_0$ to space]. let P be energy due to air pollution particulates and α be the fraction of this energy utilized in cloud formation then the energy released in form of precipitation is given by,

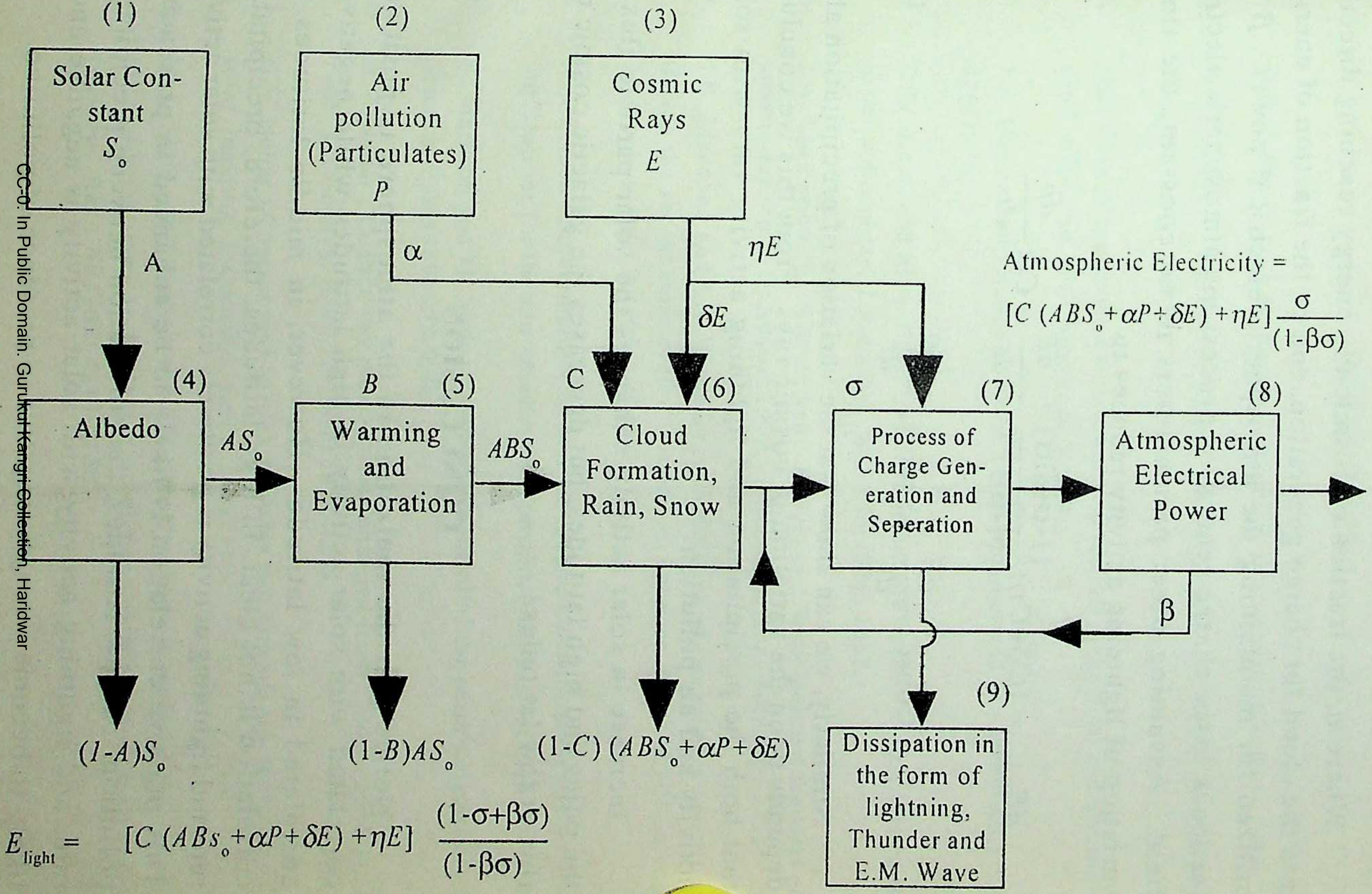
$$E_{Prec} = (1\text{-}C)\ (ABS_o+\alpha P+\delta E) \tag{1}$$

Where B is the fraction of energy which is transferred to cloud formation process, δ is the fraction of cosmic ray energy utilized in the process of nucleation (hence cloud formation), and C is the fraction of energy from cloud transferred to charge generation process. The lightning intensity can be written as (see Fig. 1)

$$E_{\text{lightn}} = [C\{ABS_0+\alpha P+\delta E) + \eta E\}]\frac{(1\text{-}\sigma+\beta\sigma)}{(1\text{-}\beta\sigma)} \tag{2}$$

Fig. 5.2 - **BLOCK DIAGRAM OF ATMOSPHERIC PROCESSES**

Fig. 5.2 - **BLOCK DIAGRAM OF ATMOSPHERIC PROCESSES**


(1)
Solar Constant
S_o
(2)
Air pollution (Particulates)
P
(3)
Cosmic Rays
E
A
α
ηE
δE
(4)
Albedo
AS_o
B
(5)
Warming and Evaporation
ABS_o
C
(6)
Cloud Formation, Rain, Snow
σ
(7)
Process of Charge Generation and Seperation
(8)
Atmospheric Electrical Power
β
(9)
Dissipation in the form of lightning, Thunder and E.M. Wave
$(1-A)S_o$
$(1-B)AS_o$
$(1-C)\,(ABS_o+\alpha P+\delta E)$

Atmospheric Electricity =

$$[C\,(ABS_o+\alpha P+\delta E)+\eta E]\,\frac{\sigma}{(1-\beta\sigma)}$$

$$E_{light} = [C\,(ABs_o+\alpha P+\delta E)+\eta E]\;\frac{(1-\sigma+\beta\sigma)}{(1-\beta\sigma)}$$

Where η, the fraction of cosmic ray energy reaching directly into the cloud for charge generation, and σ, the fraction of energy utilized for maintaining the atmospheric electrical power. β is feedback to the charge generation process via atmospheric electric field. Assuming other parameterers to be constant, the time variation of lightning activity reduces to

$$dE_{\text{lightn}} = (C\delta + C\eta)\,\frac{(1-\sigma+\beta\sigma)}{(1-\beta\sigma)}\quad\frac{dE}{dt} + C\alpha\,\frac{dp}{dt}$$

$$= \text{Const.}\ \frac{dE}{dt} + \text{const.}\ \frac{dp}{dt} \qquad (3)$$

Similarly, we can show that the variation of precipitation also depends upon the variation of cosmic rays. From this we conclude that both the precipitation and lightning activity vary with solar activity and air pollution.

Increase in solar activity enhances the solar particle flux at the poles and high latitudes, but decreases the galactic cosmic ray flux at low latitudes.

CONCLUSION

Notice that the ionization in the atmosphere is positively correlated with solar activity in high latitudes while negatively correlated in low latitudes. However, in middle latitudes the correlation is not clear. In high latitudes, therefore, precipitation rate and lightning activity is positively correlated with solar activity. This positive correlation is likely to be enhanced in presence of pollution. At low latitudes, the correlation between precipitation rate, and lightning activity with solar activity is negative and is likely to become strongly negative in presence of pollution.

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